Force reconstruction

A Bayesian perspective

Mathieu AUCEJO

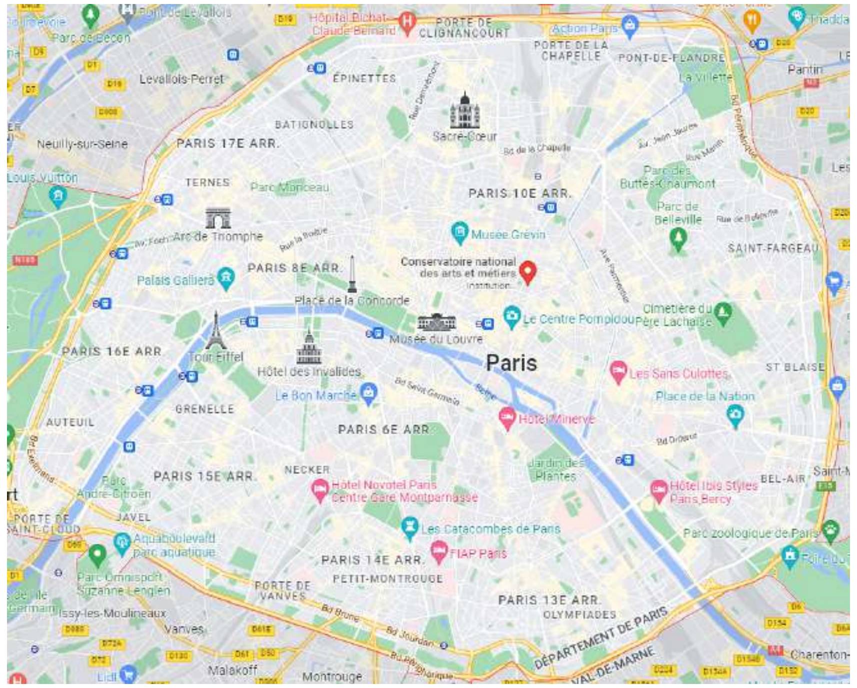
Thursday 13th October 2022

Who am I ?

It's me !

- Associate professor
- @ Le Cnam

le cnam



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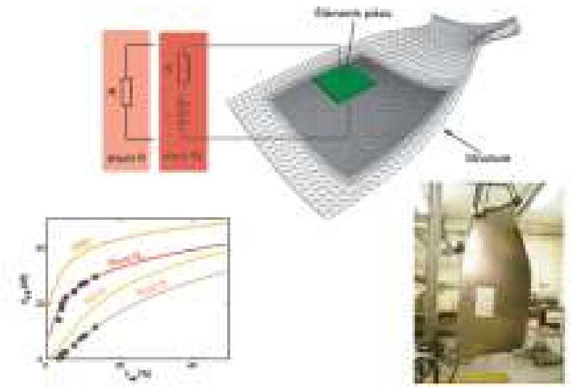
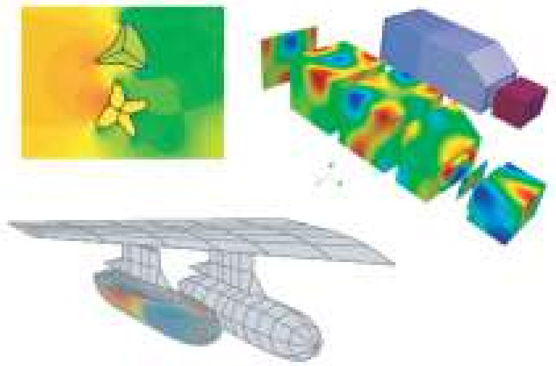
- Associate professor
- @ Le Cnam
- @ LMSSC

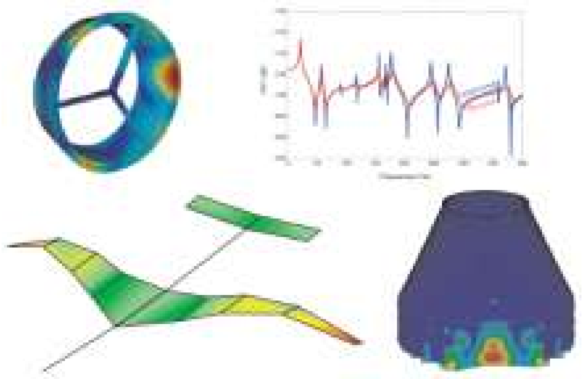
✉ mathieu.aucejo@lecnam.net

Smart structures
and adaptive interfaces

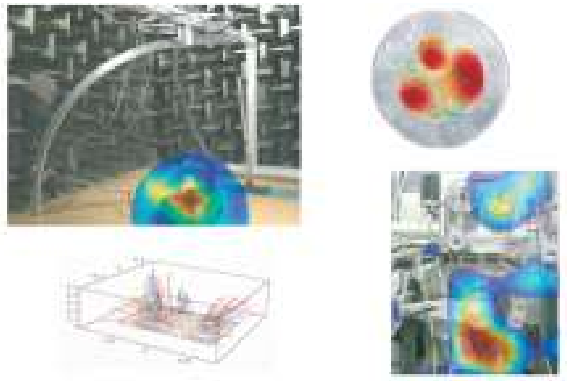


Fluid-structure interaction
and vibro-acoustics





Linear and nonlinear
structural dynamics



Source identification
and noise control

Outline

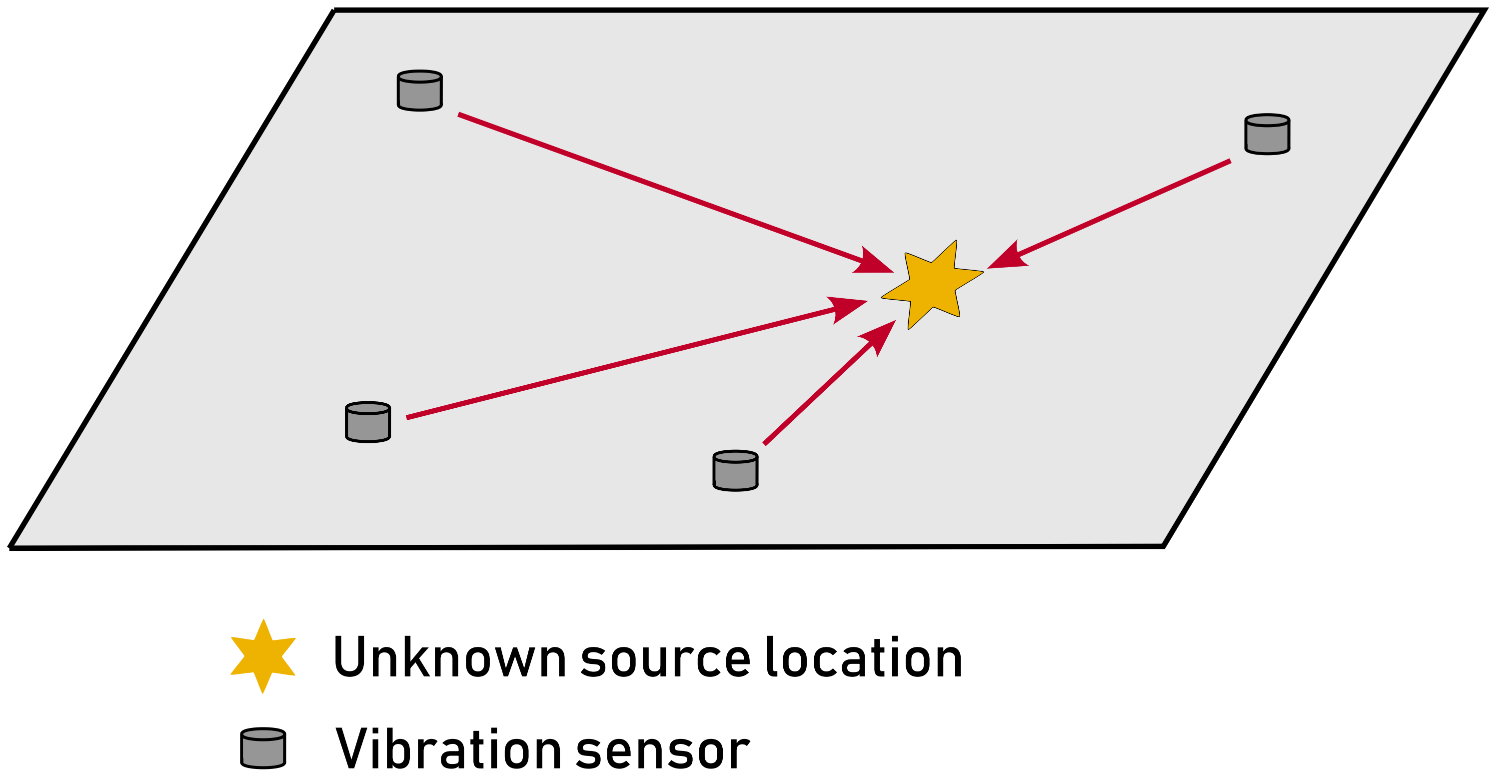
Outline

Definition

Force identification is an inverse problem aiming at characterizing some features of the sources exciting a mechanical structure

Types of problems

1. Localization

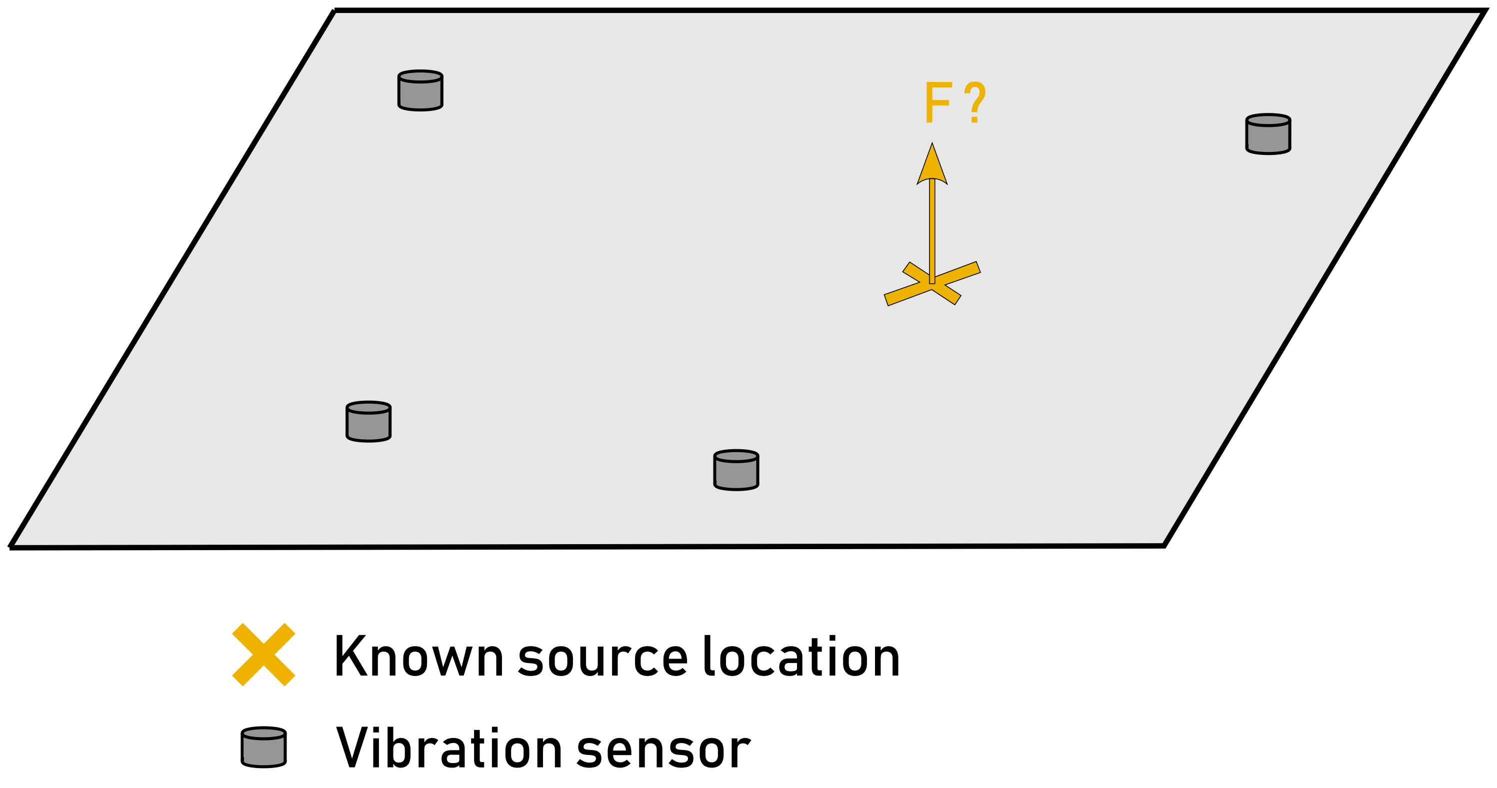


Definition

Force identification is an inverse problem aiming at characterizing some features of the sources exciting a mechanical structure

Types of problems

1. Localization
2. Quantification

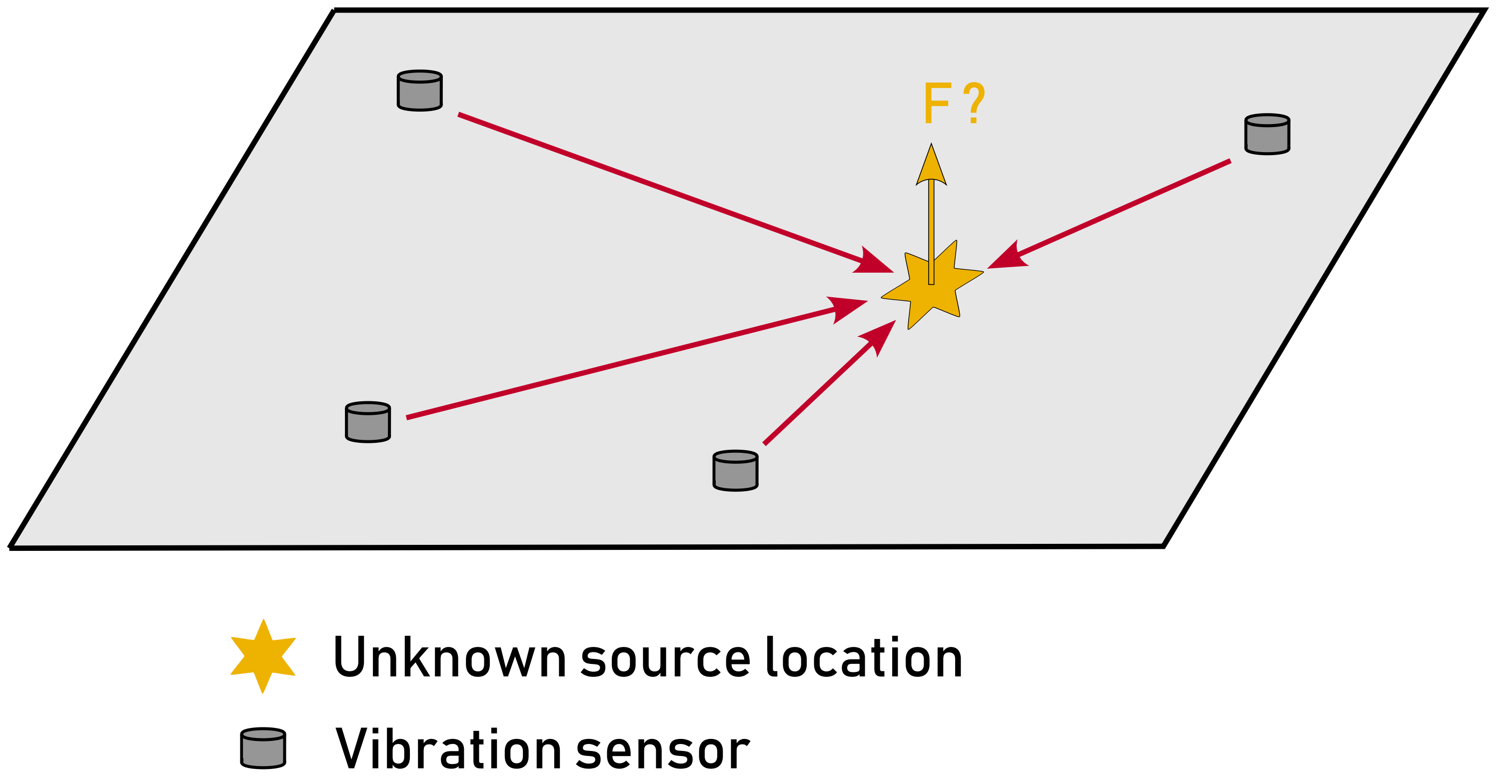


Definition

Force identification is an inverse problem aiming at characterizing some features of the sources exciting a mechanical structure

Types of problems

1. Localization
2. Quantification
3. Reconstruction

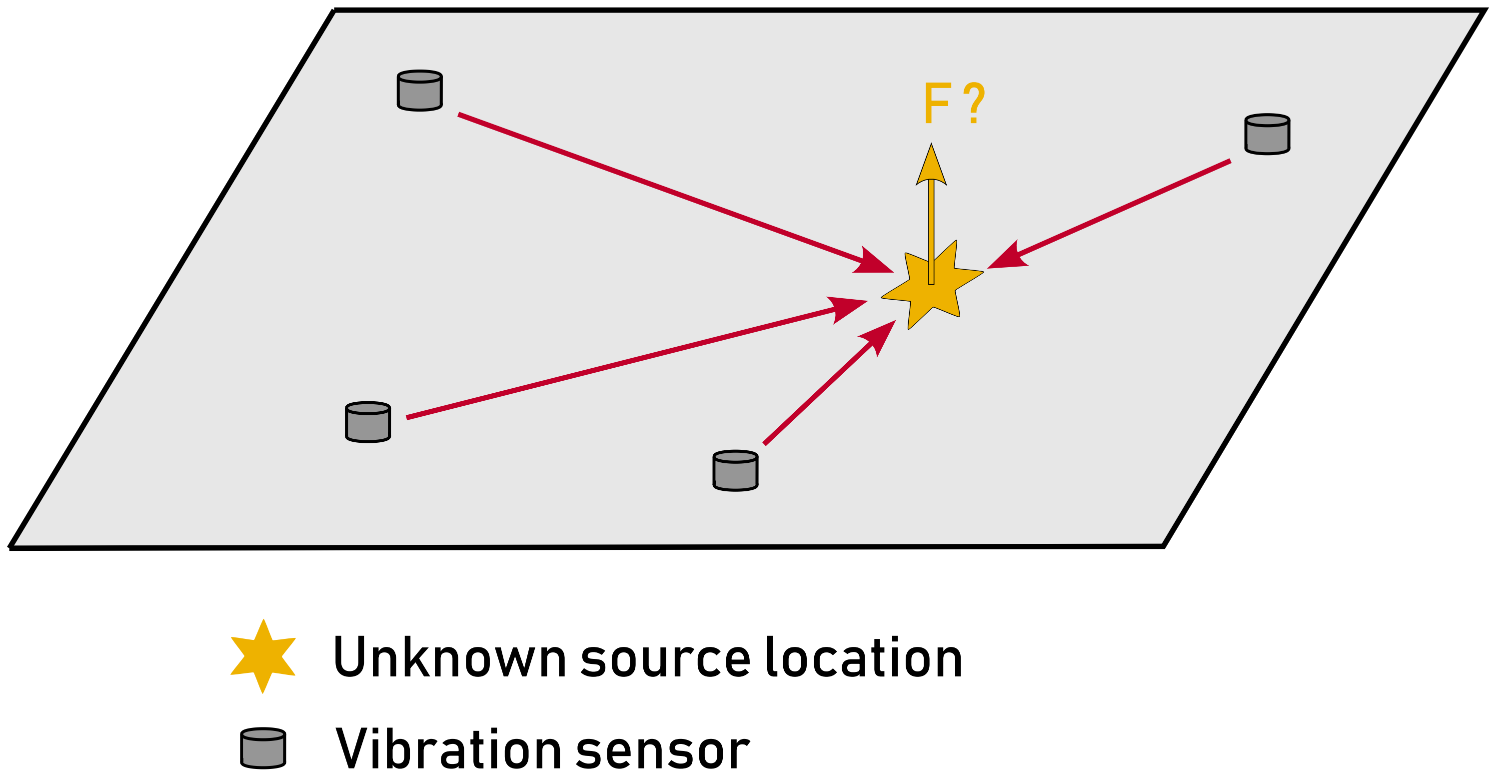


Definition

Force identification is an inverse problem aiming at characterizing some features of the sources exciting a mechanical structure

Types of problems

1. Localization
2. Quantification
3. Reconstruction
4. Separation / Classification

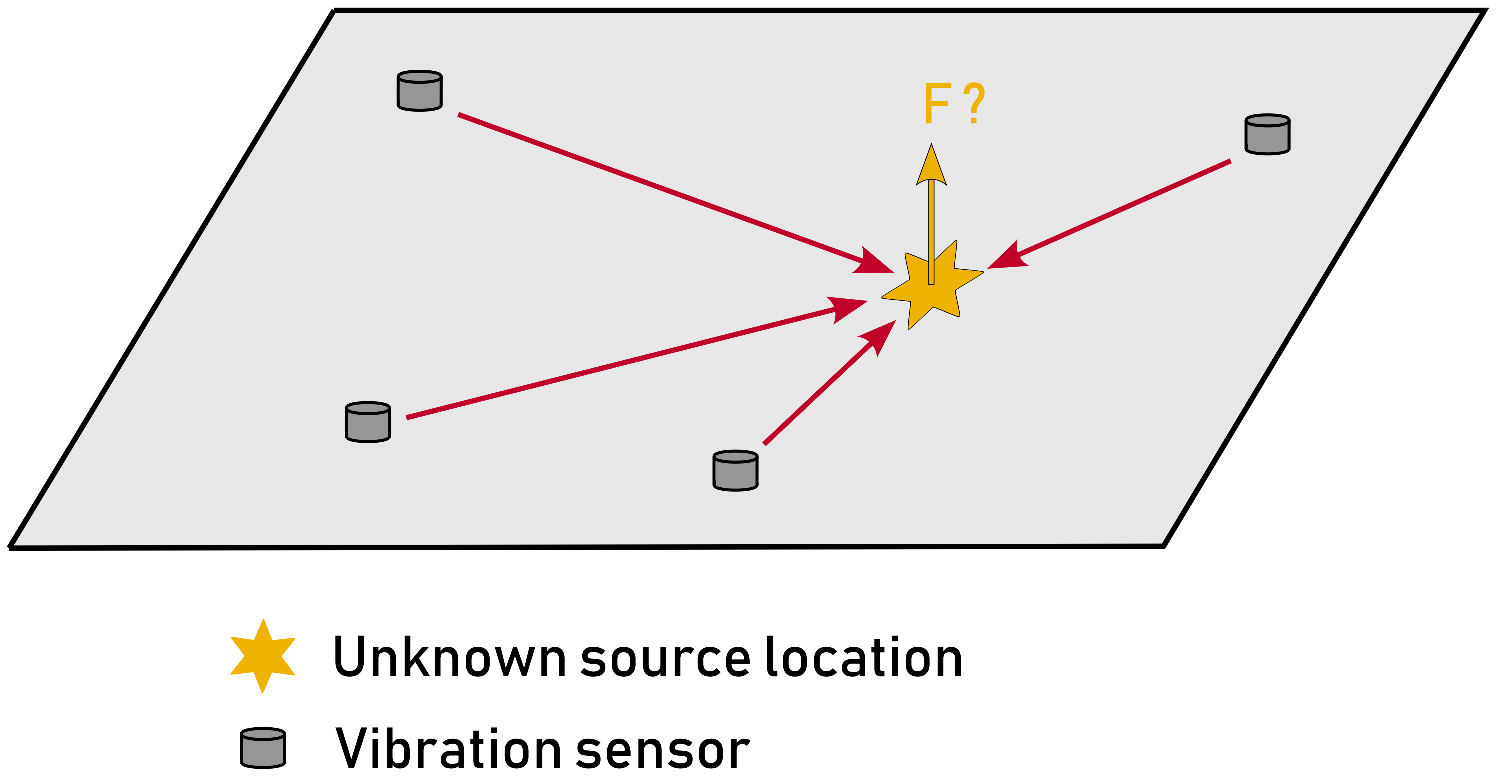


Definition

Force identification is an inverse problem aiming at characterizing some features of the sources exciting a mechanical structure

Types of problems

1. Localization
2. Quantification
3. **Reconstruction**
4. Separation / Classification



Restriction

In this lecture, we restrict ourselves to reconstruction problems expressed as a linear system

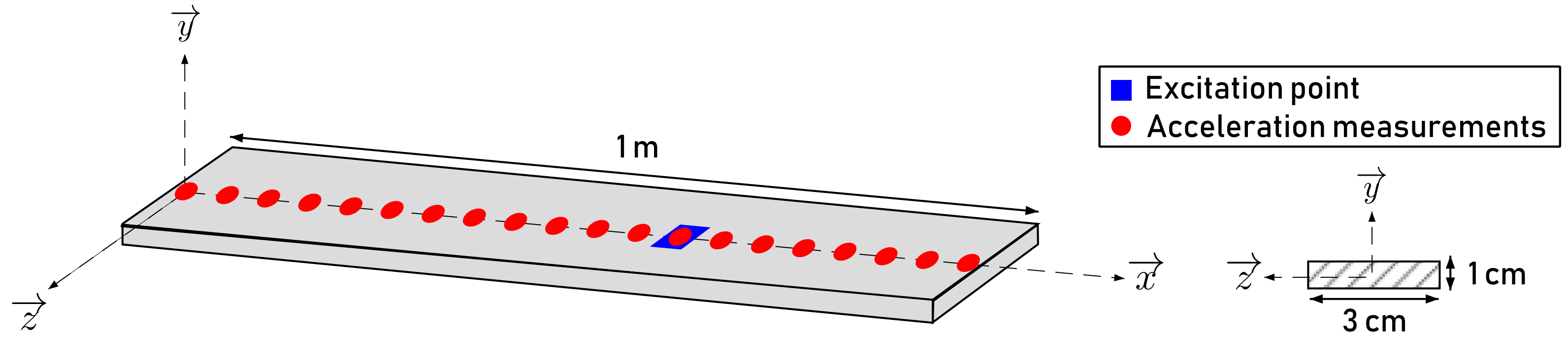
$$\mathbf{X} = \mathbf{H}\mathbf{F} + \mathbf{N}$$

- $\mathbf{X}$ is the measured vibration field
- $\mathbf{H}$ describes the dynamic behavior of the structure (LTI assumption)
- $\mathbf{F}$ is the excitation field to reconstruct
- $\mathbf{N}$ is the noise corrupting the vibration data

➥ This talk will not cover methods such as Kalman Filters, Neural Networks, Virtual Fields, ...

Outline

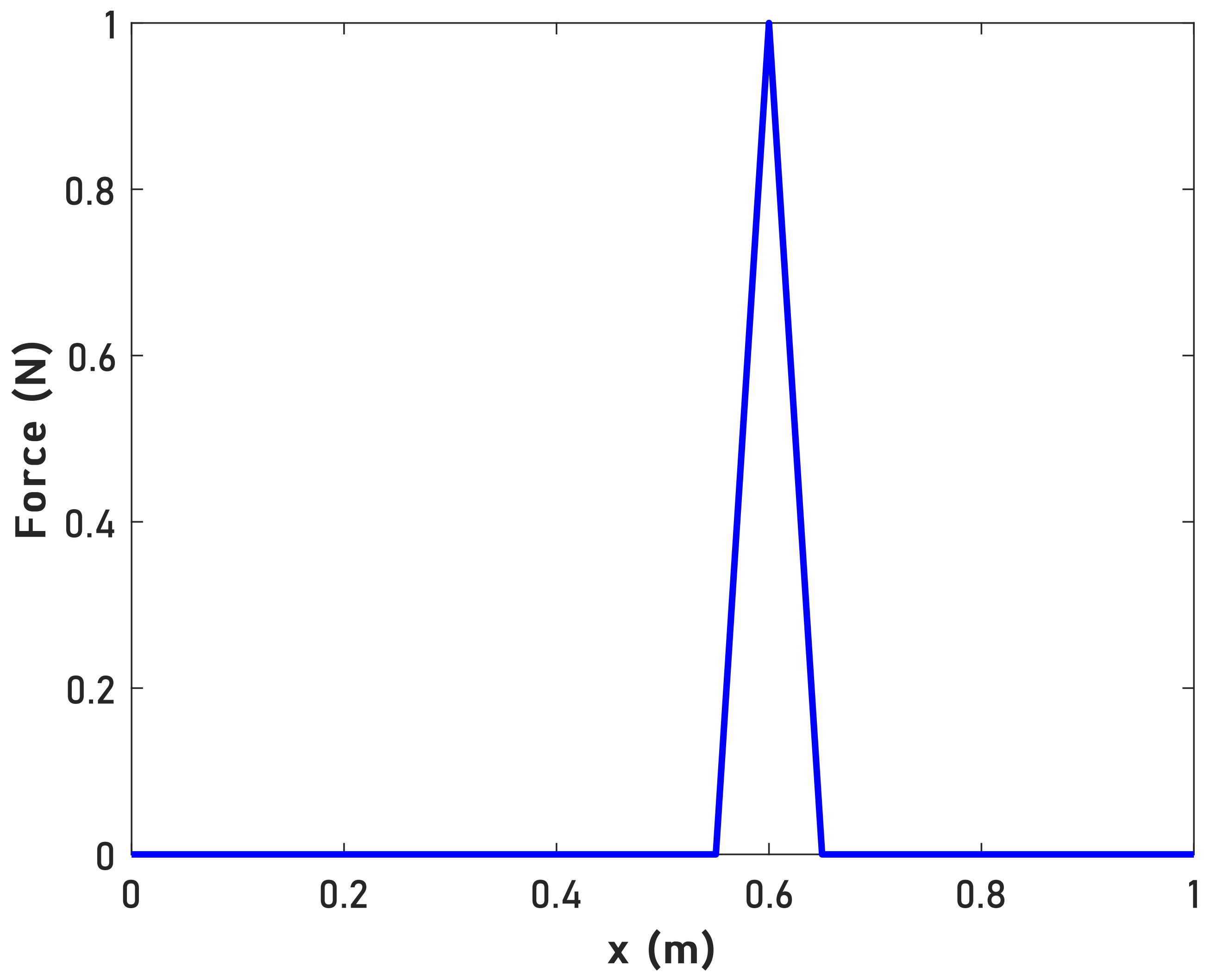
Leading example Free-free steel beam in the frequency domain



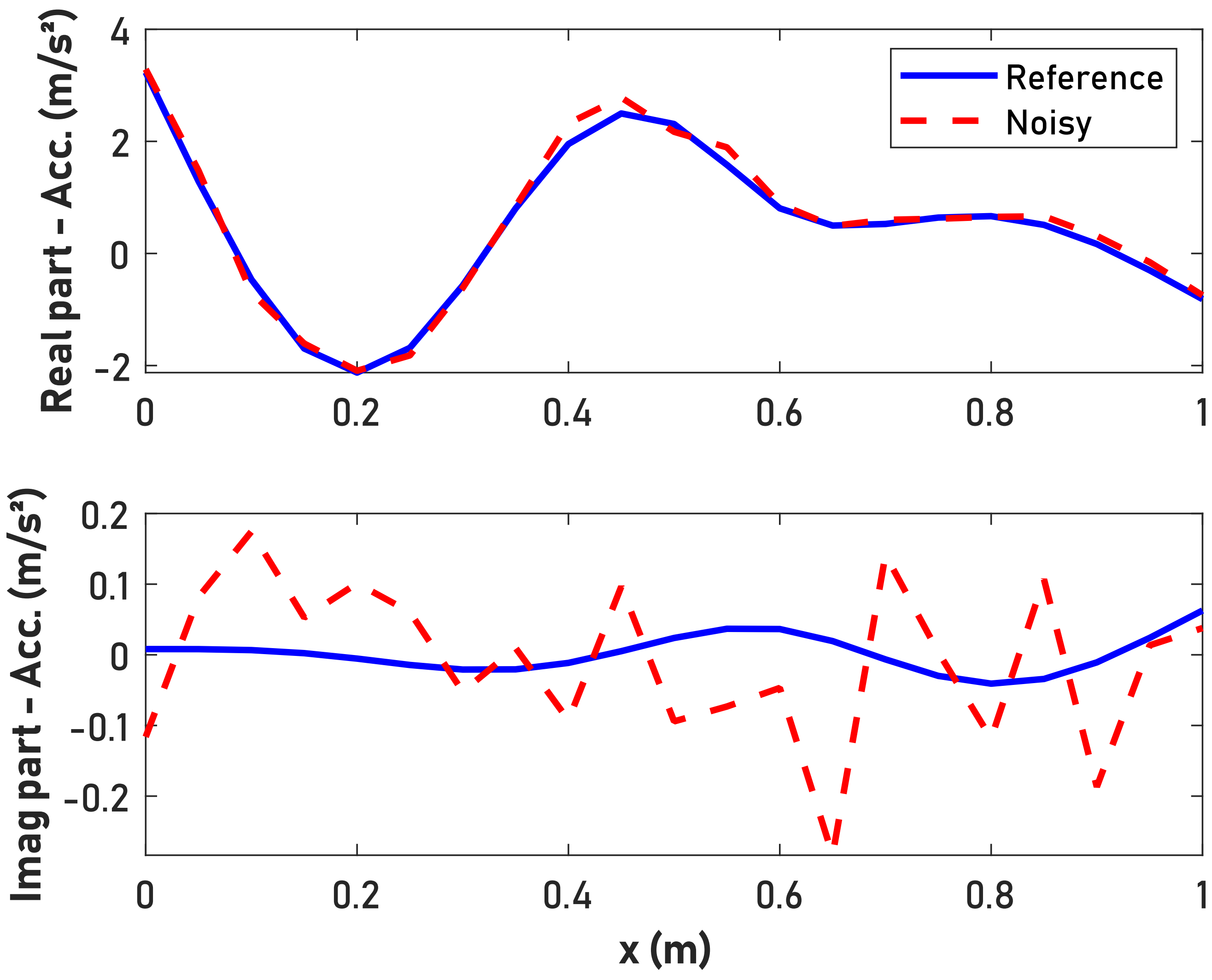
- Unit harmonic point force @ 350 Hz
- Measurement noise level - 20 dB
- Data generation - ⚠ Inverse crime
  - $\mathbf{X}$ - Modal expansion (8 modes, $f_8 \approx 992$ Hz)
  - $\mathbf{H}$ - FEM (20 beam elements)
  - Colocated reconstruction configuration
  - Equal-determined inverse problem

Main objective

Reconstruct

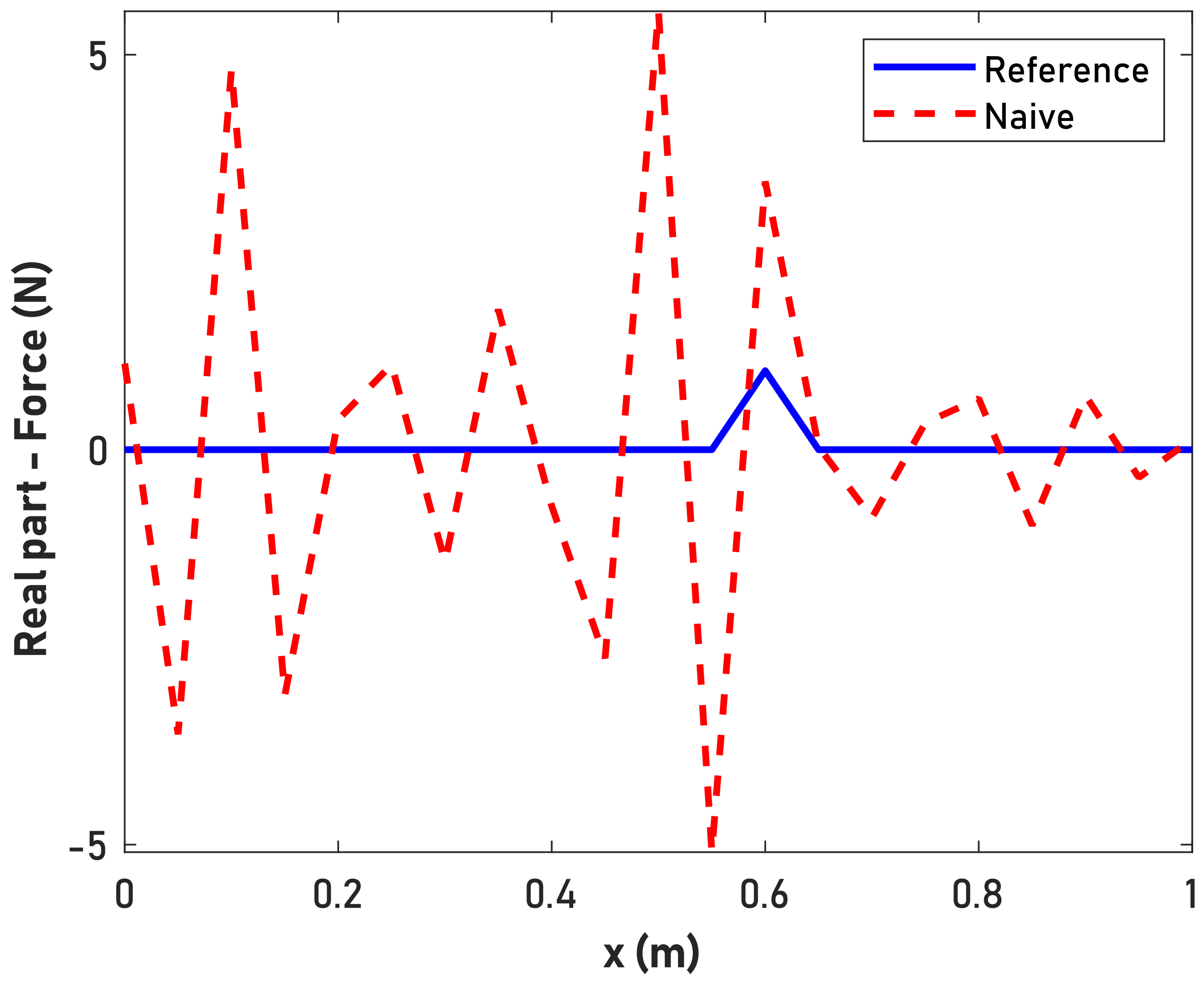


From



Naive reconstruction

$$\widehat{\mathbf{F}} = \mathbf{H}^{-1}\mathbf{X}$$



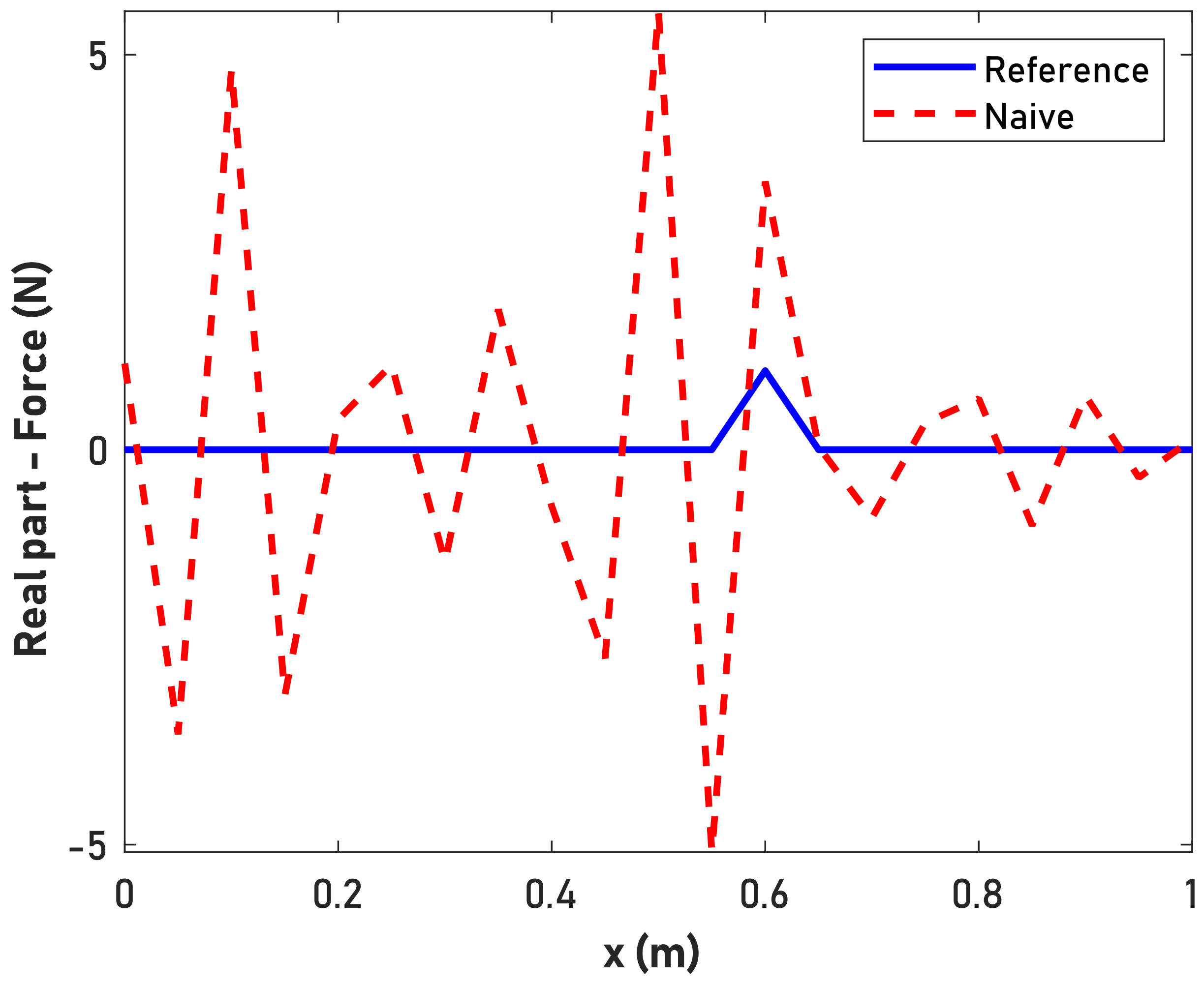
What's wrong ?

- Formally, one has:

$$\widehat{\mathbf{F}} = \sum_{i=1}^{21} \frac{\mathbf{v}_i \mathbf{u}_i^H \mathbf{X}}{\sigma_i}$$

Naive reconstruction

$$\widehat{\mathbf{F}} = \mathbf{H}^{-1}\mathbf{X}$$



What's wrong ?

- Formally, one has:

$$\widehat{\mathbf{F}} = \mathbf{F}_{\text{true}} + \sum_{i=1}^{21} \frac{\mathbf{v}_i \mathbf{u}_i^H \mathbf{N}}{\sigma_i}$$

- But $\mathbf{H}$ is ill-conditioned - $\kappa(\mathbf{H}) \approx 1300$

  Here $\sigma_{21} \approx 2.5 \cdot 10^{-2}$

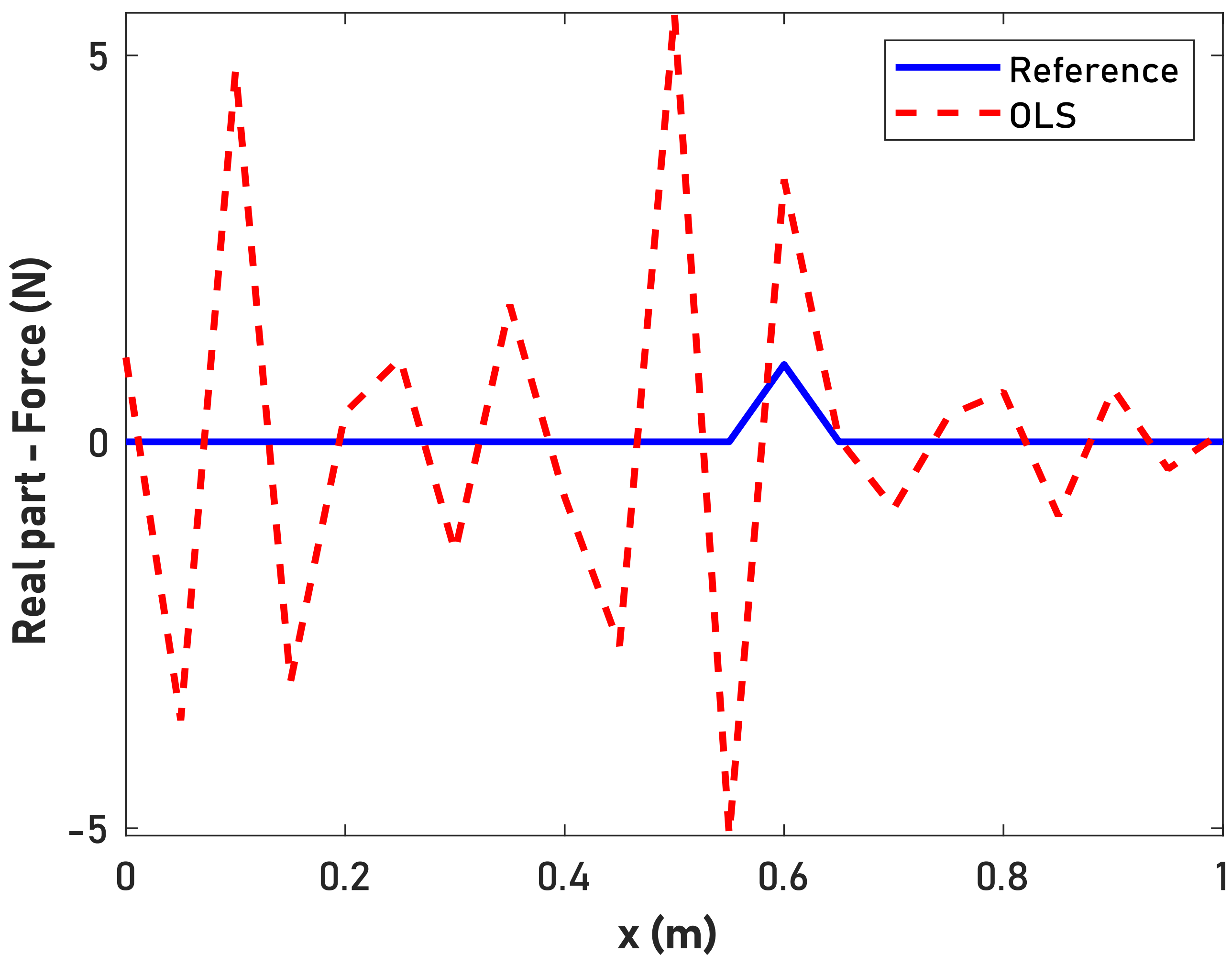
➡ The noise is amplified by the smallest singular values

➡ Ill-posed inverse problems in Hadamard sense

Ordinary Least Squares (OLS)

Idea **Find $\widehat{\mathbf{F}}$ minimizing the sum of the squared errors**

$$\widehat{\mathbf{F}} = \underset{\mathbf{F}}{\operatorname{argmin}} \; \|\mathbf{X} - \mathbf{HF}\|_2^2$$



What's wrong ?

- Formally, one has:

$$\widehat{\mathbf{F}} = (\mathbf{H}^H\mathbf{H})^{-1}\mathbf{H}^H\mathbf{X}$$

- But using the SVD

$$\widehat{\mathbf{F}} = \sum_{i=1}^{21} \frac{\mathbf{v}_i\mathbf{u}_i^H\mathbf{X}}{\sigma_i}$$

➡ Same as the naive approach ! (equal-det. problems)

➡ Useful for over/under-determined problems

Truncated SVD

Idea Filter the smallest singular values of $\mathbf{H}$

In practice Retain the first $\mathbf{M}$ singular values $(\mathbf{M} < \mathbf{21})$ such that

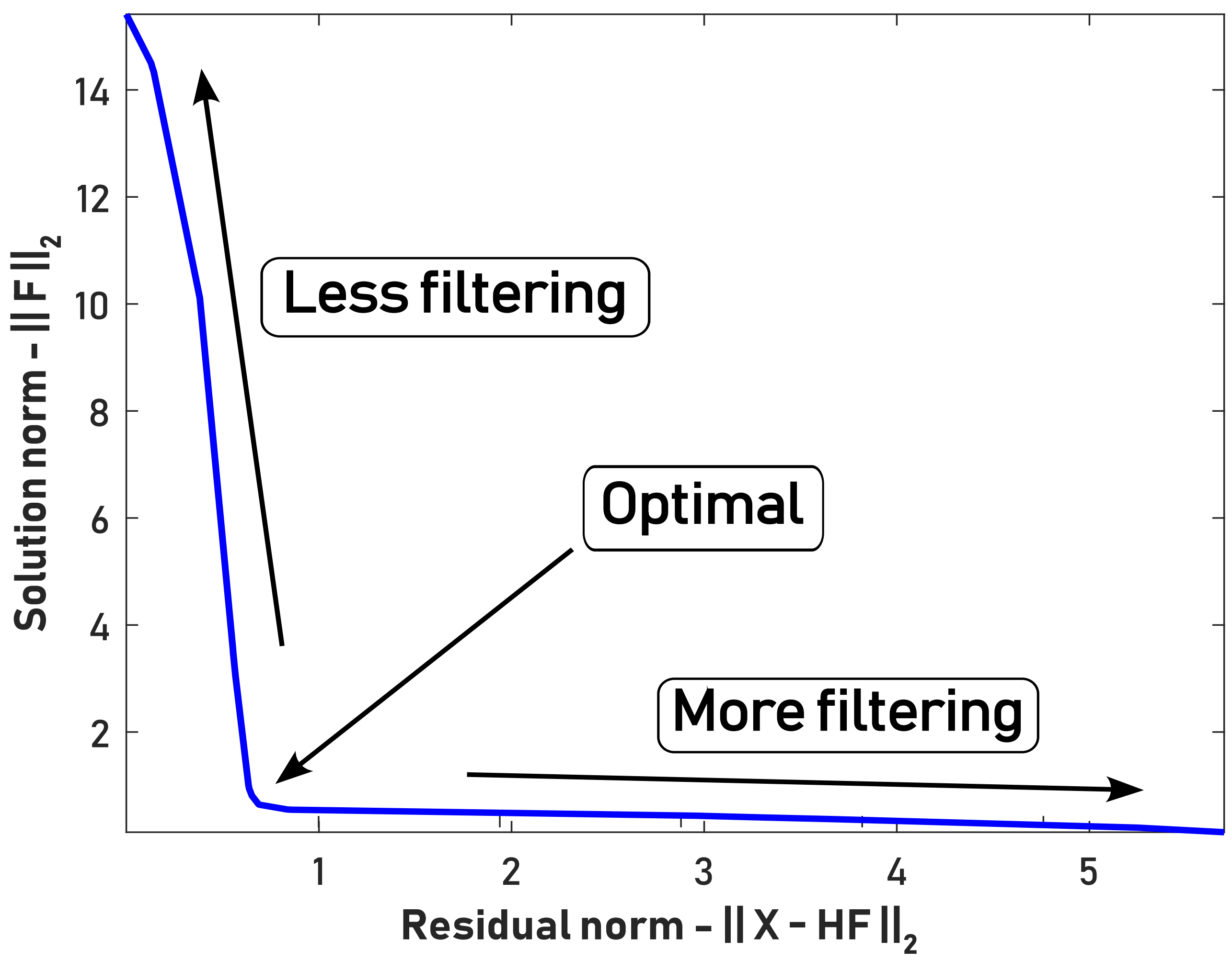
$$\widehat{\mathbf{F}} = \sum_{i=1}^{M} \frac{\mathbf{v}_i \mathbf{u}_i^H \mathbf{X}}{\sigma_i}$$

How to select M ?

One possible solution L-curve principle

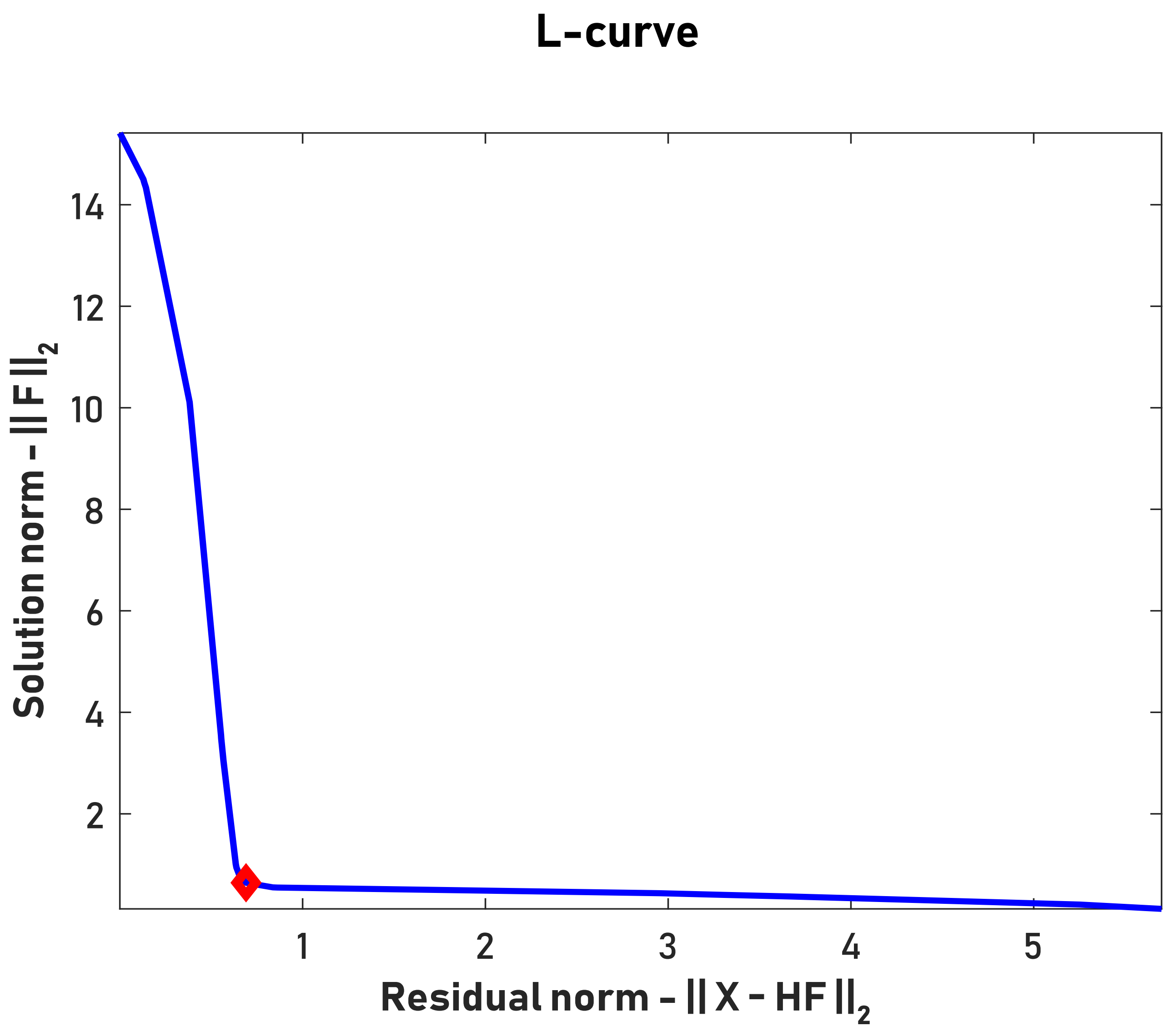
$$L_c(M) = \left(\|\mathbf{X} - \mathbf{H}(M)\widehat{\mathbf{F}}\|_2,\ \|\widehat{\mathbf{F}}\|_2\right) \text{ with } \mathbf{H}(M) = \sum_{i=1}^{M} \sigma_i \mathbf{u}_i \mathbf{v}_i^H$$

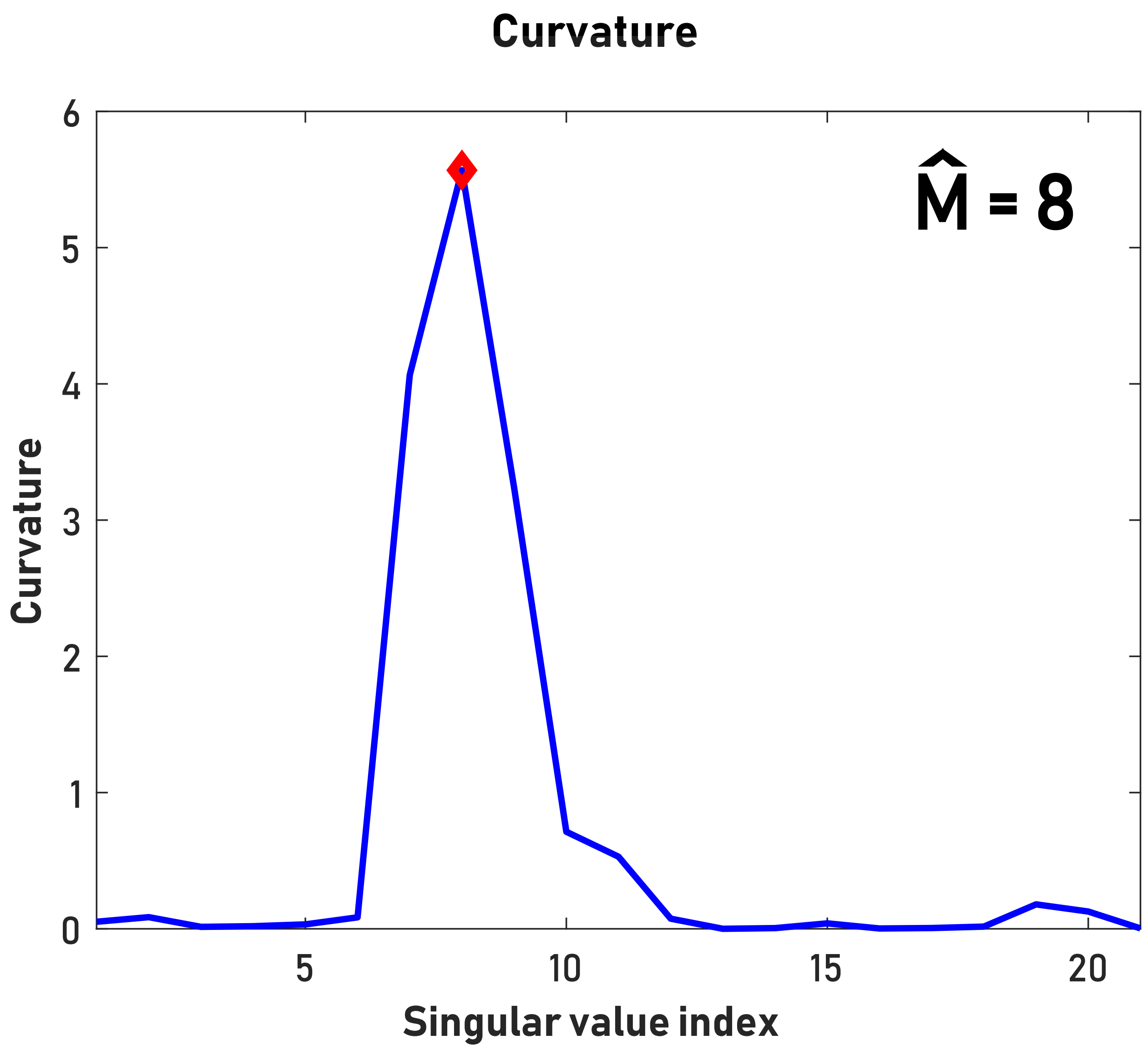
L-curve



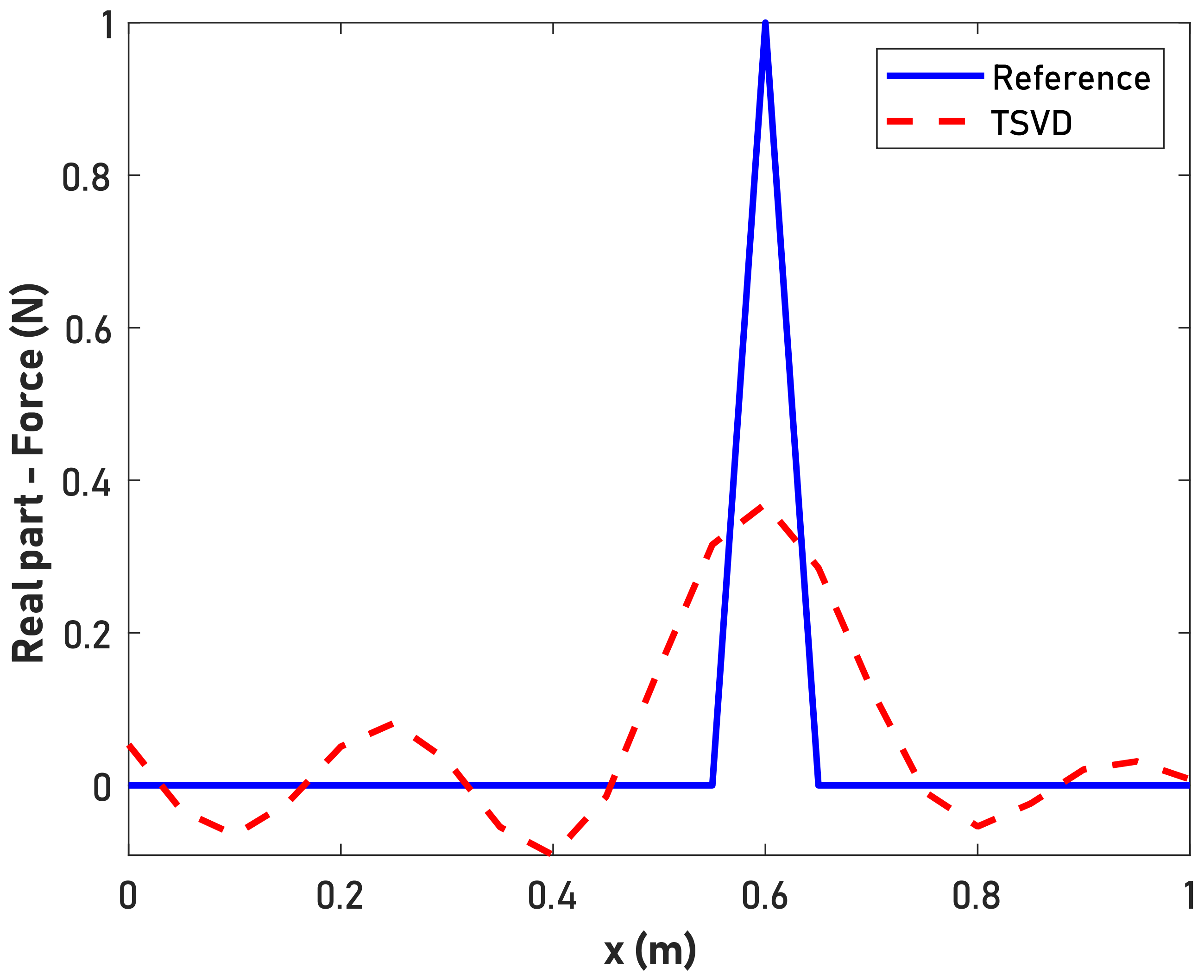
One possible solution L-curve principle

$$\widehat{M} = \underset{M}{\operatorname{argmax}} \; K[L(M)]$$





Application



- Low pass filtering effect ⇒ Smooth solution

➡ Not adapted to sparse sources

What to do ?

Constrain the space of admissible solutions !

ℓ_2-regularization Tikhonov regularization

$$\widehat{\mathbf{F}} = \underset{\mathbf{F}}{\operatorname{argmin}} \;\; \|\mathbf{X} - \mathbf{HF}\|_2^2 \;\; \text{subject to} \;\; \|\mathbf{F}\|_2^2 \leq \tau$$

ℓ_2-regularization Tikhonov regularization

$$\widehat{\mathbf{F}} = \underset{\mathbf{F}}{\operatorname{argmin}} \; \|\mathbf{X} - \mathbf{HF}\|_2^2 + \lambda \|\mathbf{F}\|_2^2$$

How to select λ ?

In practice Many methods are available

- Morozov's discrepancy principle
- Generalized Cross Validation (GCV)
- Reginska's method
- Bayesian Estimator
- L-curve principle
- ....

ℓ_2-regularization Tikhonov regularization

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- Morozov's discrepancy principle
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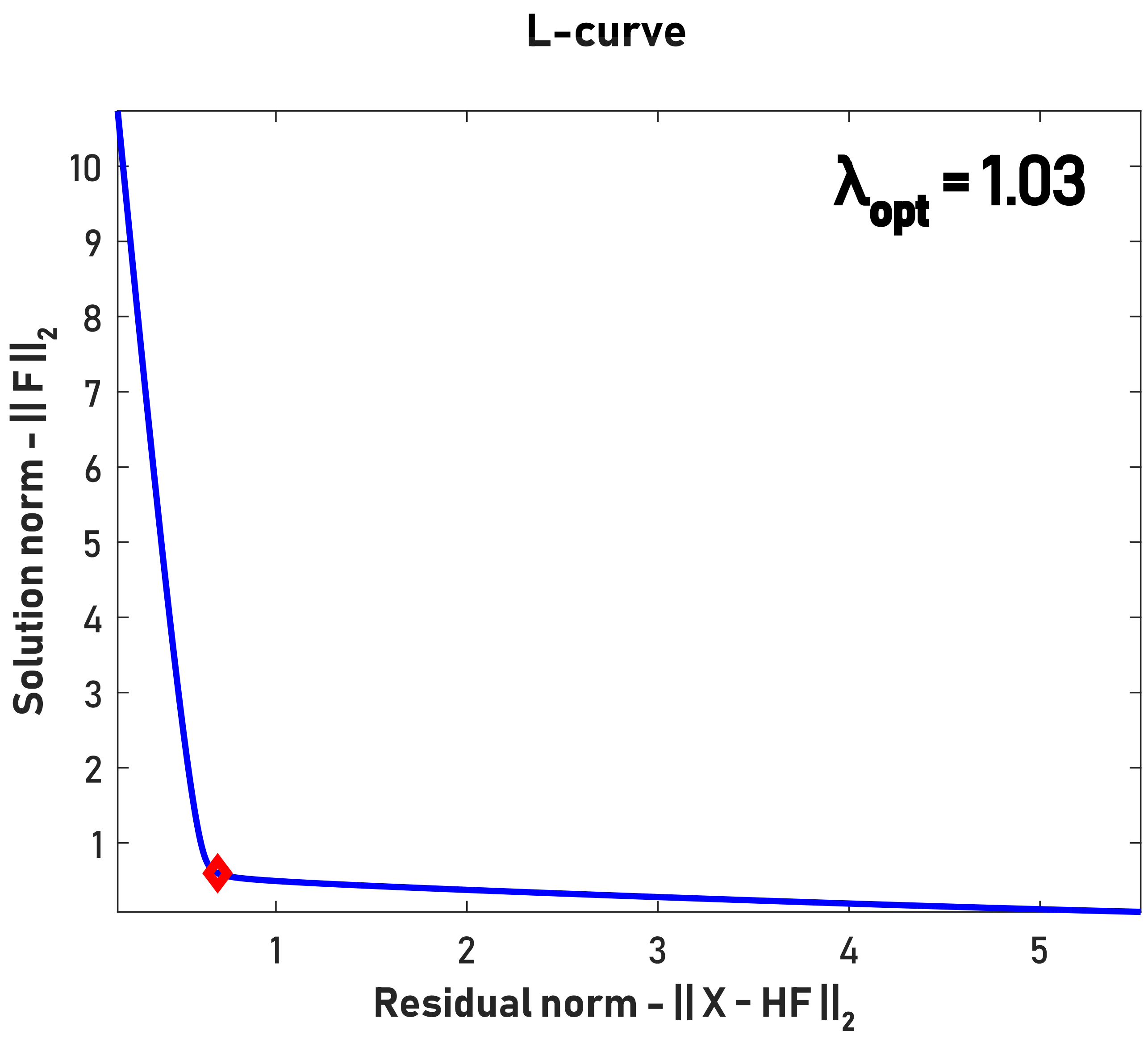
- Bayesian Estimator
- **L-curve principle**
- ....

ℓ_2-regularization Application

$$\widehat{\mathbf{F}} = (\mathbf{H}^H\mathbf{H} + \lambda\mathbf{I})^{-1}\mathbf{H}^H\mathbf{X}$$



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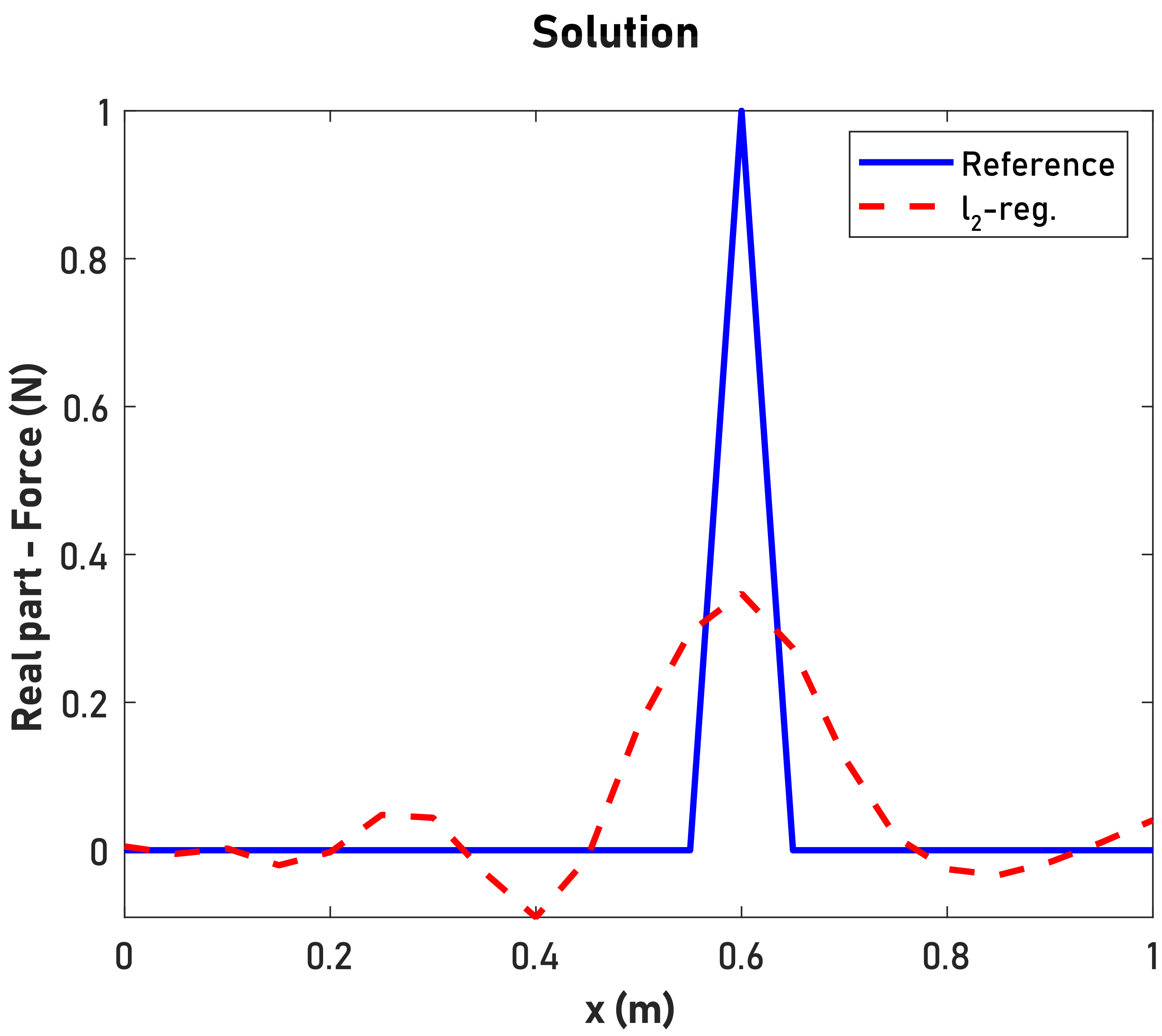


- Low pass filtering effect ⇒ Smooth solution

➡ Not adapted to sparse sources

How to explain this result ?

Filter factors Basics

$$\widehat{\mathbf{F}} = \sum_{i=1}^{21} f_i \frac{\mathbf{v}_i \mathbf{u}_i^H \mathbf{X}}{\sigma_i}$$

where f_i is the filter factor defined such that

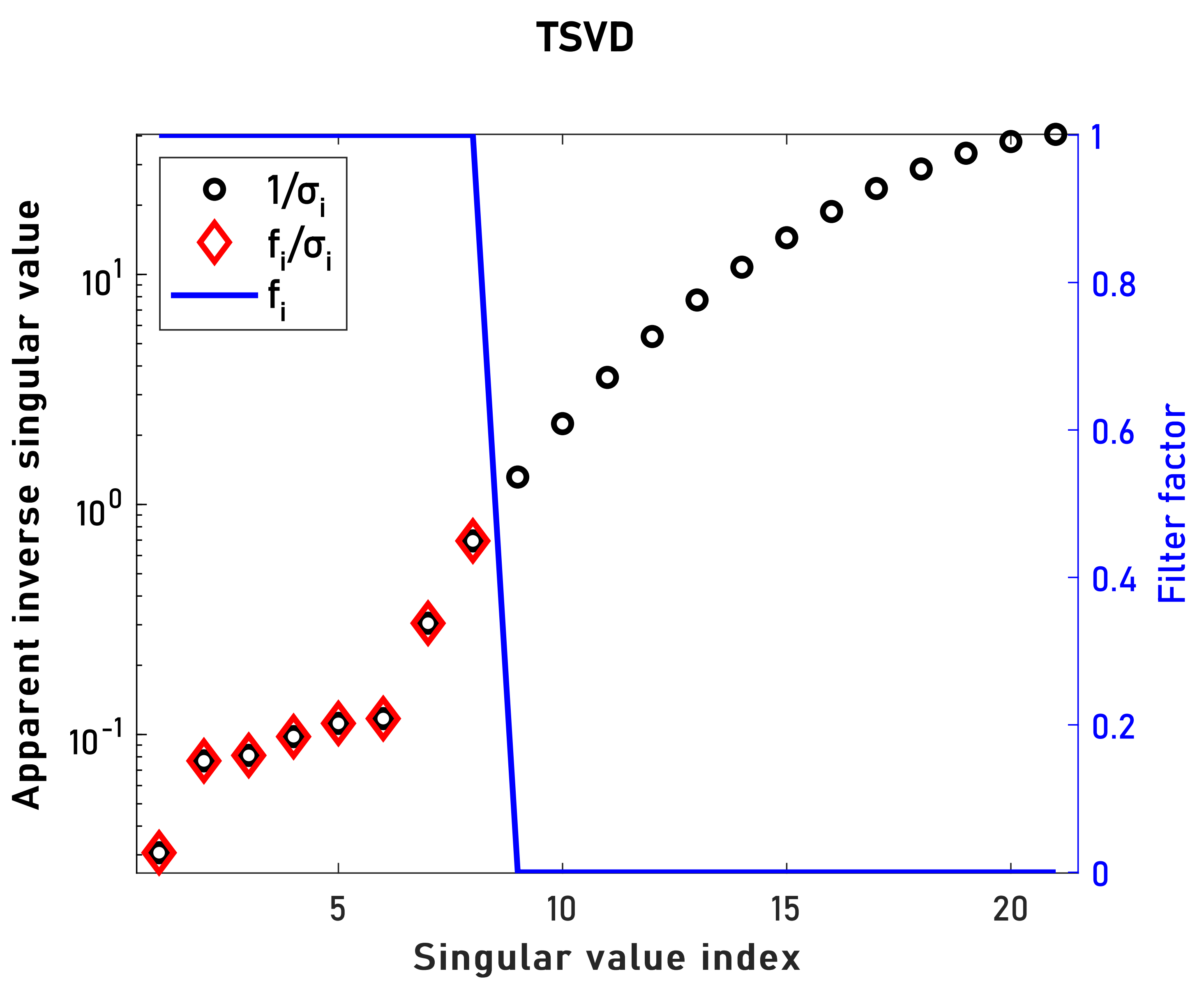
TSVD

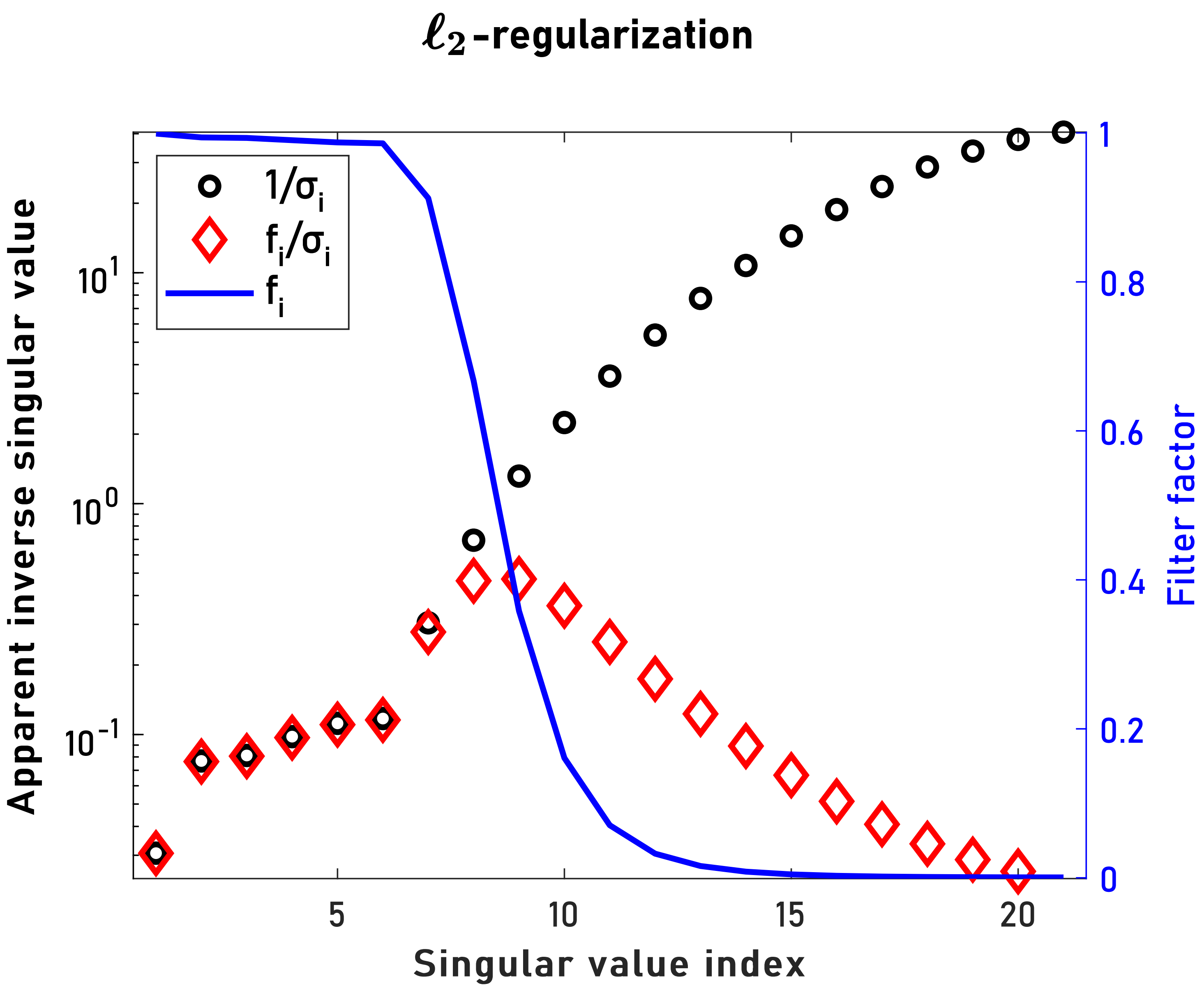
$$f_i = \begin{cases} 1 & \text{for } i \leq M \\ 0 & \text{otherwise} \end{cases}$$

ℓ_2-regularization

$$f_i = \frac{\sigma_i^2}{\sigma_i^2 + \lambda}$$

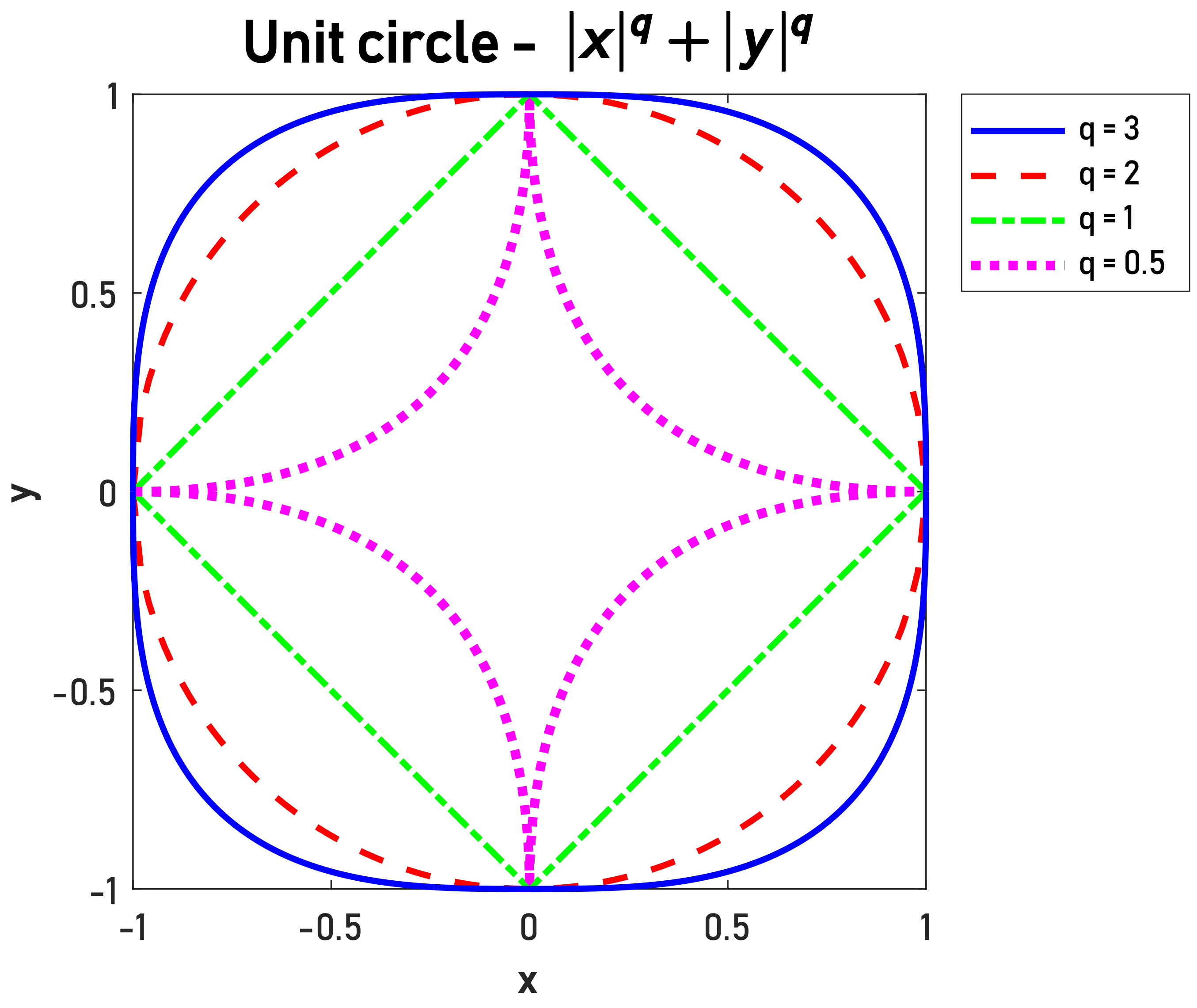
Filter factors In action





ℓ_q-regularization Generalities

$$\widehat{\mathbf{F}} = \underset{\mathbf{F}}{\operatorname{argmin}} \; \|\mathbf{X} - \mathbf{H}\mathbf{F}\|_2^2 + \lambda \|\mathbf{F}\|_q^q$$



- The smaller q is, the larger is the weight on small values of $\mathbf{F}$
- For large values of $\mathbf{F}$, the smaller q is, the smaller is the weight on these values

➡ $q \geq 2$ - Smooth solution

➡ $q \leq 1$ - Sparse solution

⚠ Non-convex minimization problem when $q < 1$

ℓ_q-regularization Numerical resolution

The first-order optimality condition for the ℓ_q-regularization leads to

$$\widehat{\mathbf{F}} = \left(\mathbf{H}^H\mathbf{H} + \lambda\mathbf{W}(\widehat{\mathbf{F}})\right)^{-1}\mathbf{H}^H\mathbf{X} \quad \text{with} \quad w_{ii} = \frac{q}{2}\left|\widehat{F}_i\right|^{q-2}$$

➥ Implementation of an iterative process

$$\widehat{\mathbf{F}}^{(k)} = \left(\mathbf{H}^H\mathbf{H} + \lambda^{(k)}\mathbf{W}\left(\widehat{\mathbf{F}}^{(k-1)}\right)\right)^{-1}\mathbf{H}^H\mathbf{X}$$

ℓ_q-regularization Numerical resolution

The first-order optimality condition for the ℓ_q-regularization leads to

$$\widehat{\mathbf{F}} = \left(\mathbf{H}^H\mathbf{H} + \lambda\mathbf{W}(\widehat{\mathbf{F}})\right)^{-1}\mathbf{H}^H\mathbf{X} \text{ with } w_{ii} = \frac{q}{2}\left|\widehat{F}_i\right|^{q-2}$$

➡ Implementation of an iterative process

$$\widehat{\mathbf{F}}^{(k)} = \underset{\mathbf{F}}{\operatorname{argmin}} \|\mathbf{X} - \mathbf{H}\mathbf{F}\|_2^2 + \lambda^{(k)}\|\mathbf{L}\mathbf{F}\|_2^2 \text{ with } \mathbf{W}\left(\widehat{\mathbf{F}}^{(k-1)}\right) = \mathbf{L}^H\mathbf{L}$$

where $\lambda^{(k)}$ is selected from the following L-curve

$$L_c(\lambda^{(k)}) = \left(\|\mathbf{X} - \mathbf{H}\mathbf{F}(\lambda^{(k)})\|_2, \|\mathbf{L}\mathbf{F}(\lambda^{(k)})\|_2\right)$$

When the iterative process has converged, one has

$$\|\mathbf{L}\widehat{\mathbf{F}}\|_2^2 \approx \|\widehat{\mathbf{F}}\|_q^q$$

ℓ_q-regularization Practical implementation

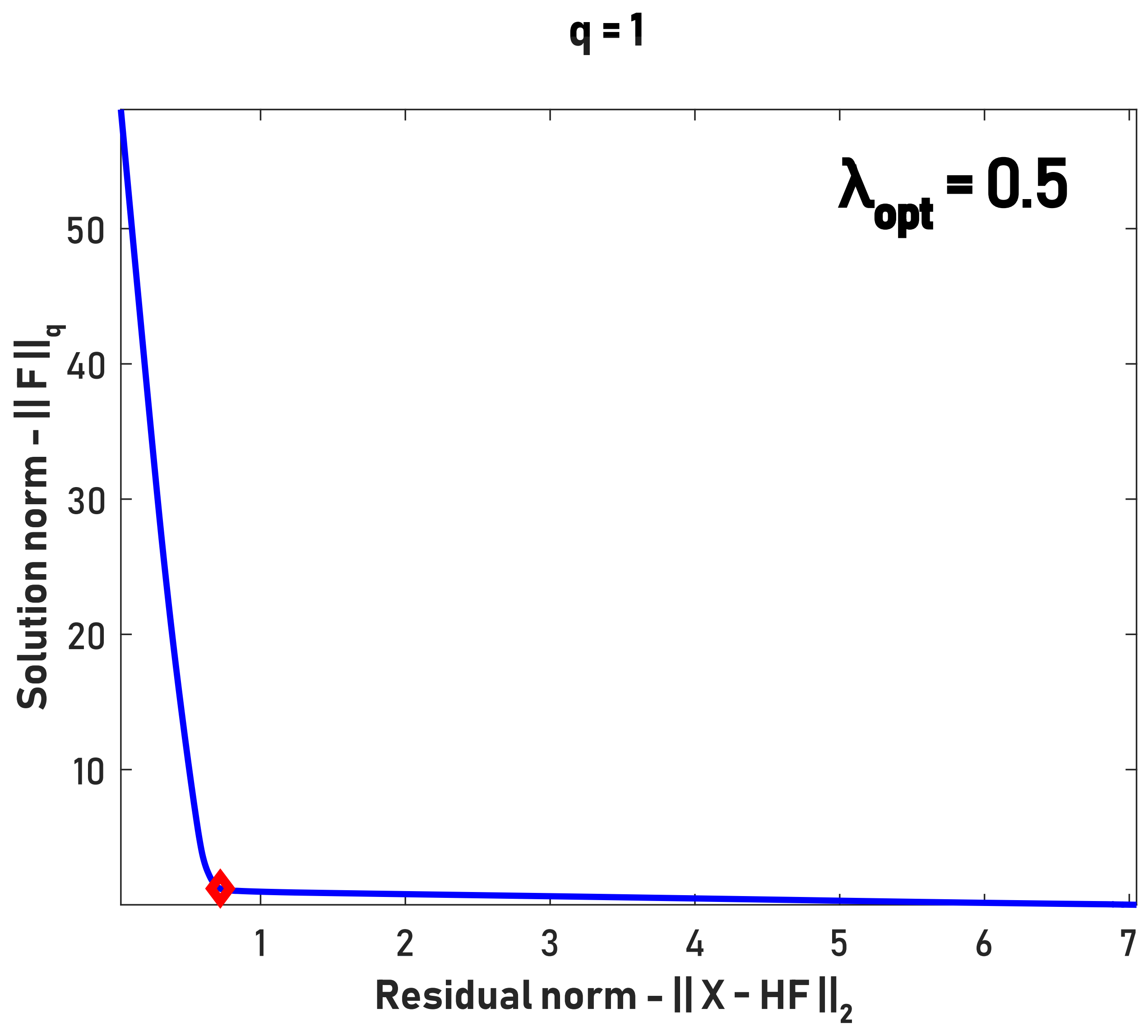
Matlab

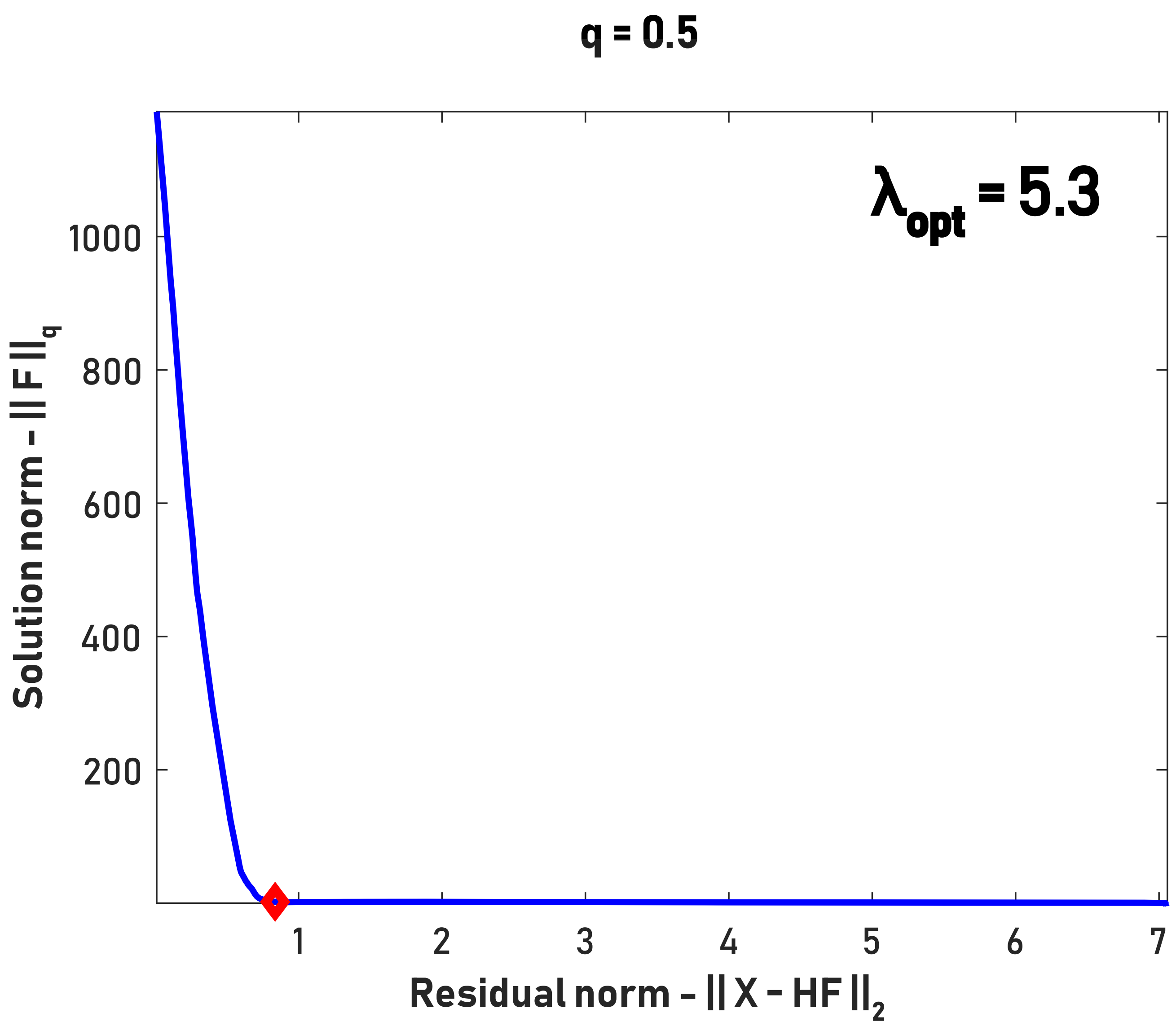
```
function [F, lamb] = lq_reg(H, X, q, tol)

% Initialization
N = size(H, 2)
Hh = H'*H; % For speed
Hx = H'*X;

L = eye(N)
lamb = lcurve(H, L, X);
F = (Hh + lamb*L)\(Hx);
F0 = F; % For convergence monitoring

% Iteration
crit = 1; % Convergence criterion
while crit > tol
        W = weight(F, q);
        L = sqrt(W) % W = L'*L;
        lamb = lcurve(H, L, X);
        F = (Hh + lamb*W)\Hx;

        % Convergence monitoring
        crit = norm(F - F0, 1)/norm(F0, 1);
        F0 = F;
end
```

Python

```
def lq_reg(H, X, q, tol):

        # Initialization
        N = H.shape[1]
        Hh = H.T.conj() @ H # For speed
        Hx = H.T.conj() @ X

        L = np.eye(N)
        lamb = lcurve(H, L, X)
        F = spl.solve(Hh + lamb*L, Hx)
        F0 = F # For convergence monitoring

        # Iteration
        crit = 1 # Convergence criterion
        while crit > tol:
                W = weight(F, q)
                L = np.sqrt(W) # W = L.T.conj()*L;
                lamb = lcurve(H, L, X)
                F = spl.solve(Hh + lamb*W, Hx)

                # Convergence monitoring
                crit = spl.norm(F - F0, 1)/spl.norm(F0, 1)
                F0 = F

        return F, lamb
```

ℓ_q-regularization Sparse regularization





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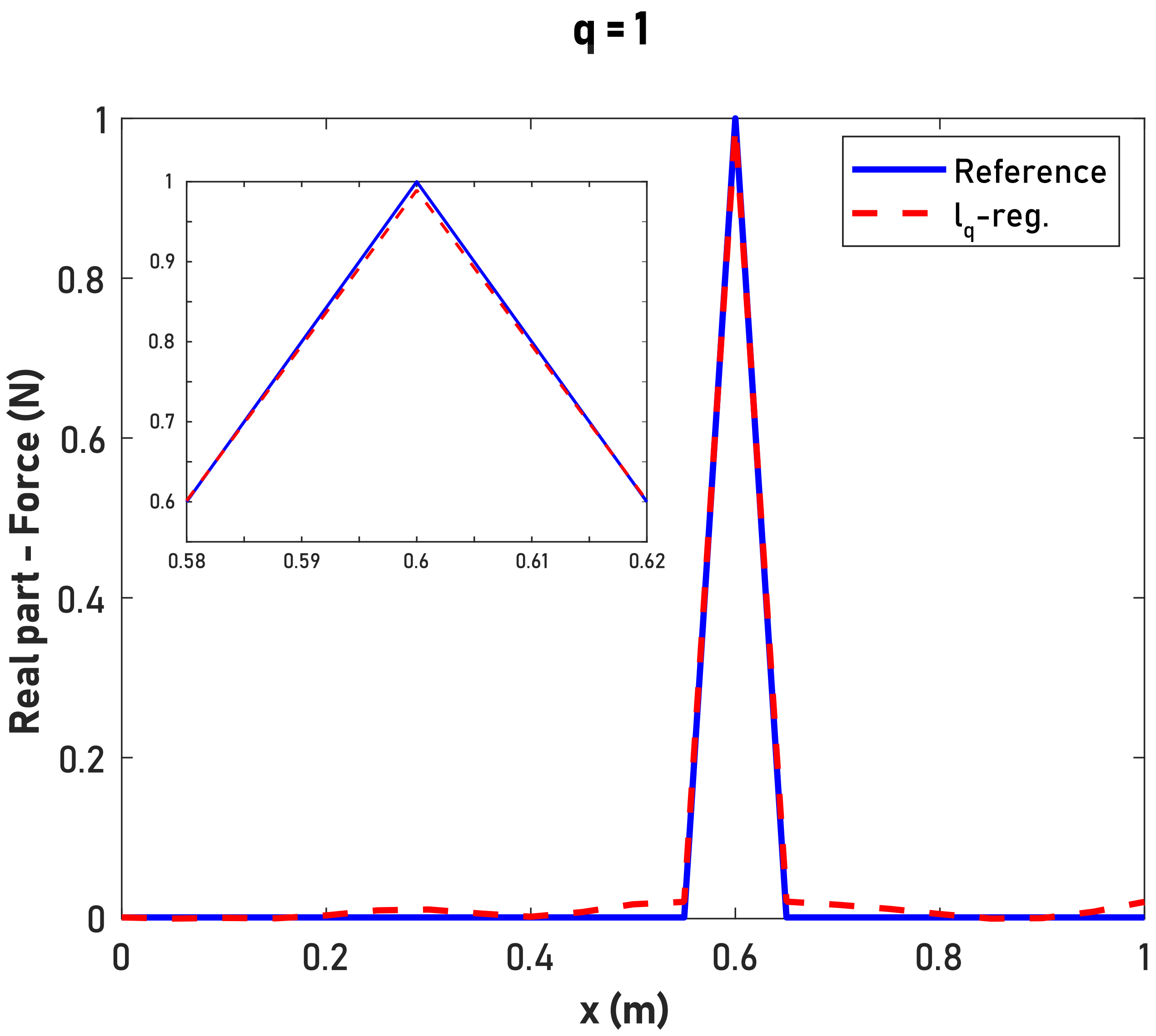
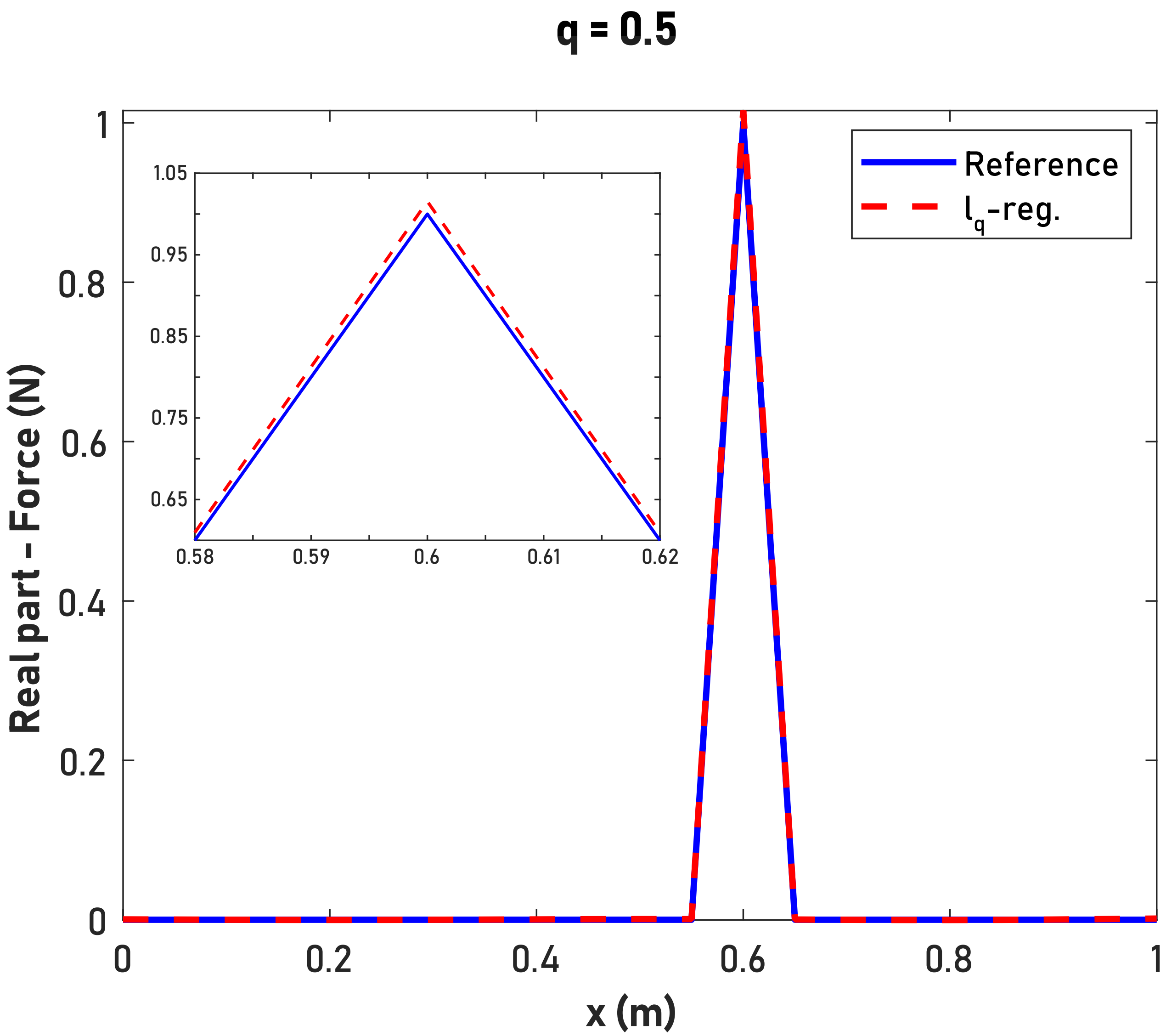




Filter factor analysis

Summary of regularization strategies

- ✓ Efficient approaches
- ✓ Easy implementation of resolution algorithms

But...

- ~ Require external procedures to determine the regularization parameter
- ~ Provide only point estimate ⇒ No uncertainty quantification of identified solutions

Possible solution ?

Summary of regularization strategies

✓ Efficient approaches

✓ Easy implementation of resolution algorithms

But...

~ Require external procedures to determine the regularization parameter

~ Provide only point estimate $\Rightarrow$ No uncertainty quantification of identified solutions

Exploit the Bayesian paradigm !

Outline

Preliminaries Bayes' rule (1763 – posthumously)

For two events A and B

$$p(A|B) \propto p(B|A)\, p(A)$$

- $p(A|B)$ – **Posterior probability distribution**
  probability of A given a realization of B
- $p(B|A)$ – **Likelihood function**
  probability of B given a realization of A
- $p(A)$ – **Prior probability distribution**
  probability of A without any given conditions

The Bayes' rule updates our prior belief in A considering new information brought by an event B

Minimal formulation **Basics**

When choosing $A = \mathbf{F}$ and $B = \mathbf{X}$

$$p(\mathbf{F}|\mathbf{X}) \propto p(\mathbf{X}|\mathbf{F})\ p(\mathbf{F})$$

How to choose $p(\mathbf{X}|\mathbf{F})$ and $p(\mathbf{F})$?

Minimal formulation Likelihood function

The likelihood function describes the probability of the observed data as a function of the parameters of the chosen statistical model. Given our linear model $\mathbf{X} = \mathbf{HF} + \mathbf{N}$, it reflects the uncertainty related to vibration measurements, i.e. related to measurement noise

Main assumption

The noise is due to multiple independent causes ⇒ **Gaussian white noise**

$$p(\mathbf{X}|\mathbf{F}, \tau_n) = \mathcal{N}_c(\mathbf{X}|\mathbf{HF}, \tau_n^{-1}\,\mathbf{I})$$

Minimal formulation Likelihood function

The likelihood function describes the probability of the observed data as a function of the parameters of the chosen statistical model. Given our linear model $\mathbf{X} = \mathbf{HF} + \mathbf{N}$, it reflects the uncertainty related to vibration measurements, i.e. related to measurement noise

Main assumption

The noise is due to multiple independent causes ⇒ **Gaussian white noise**

$$p(\mathbf{X}|\mathbf{F}, \tau_n) = \left(\frac{\tau_n}{\pi}\right)^N \exp\left(-\tau_n \|\mathbf{X} - \mathbf{HF}\|_2^2\right)$$

- τ_n - Noise precision ($\tau_n > 0$)
- N - Number of measurement points

Minimal formulation Prior probability distribution

The prior probability distribution reflects the uncertainty related to $\mathbf{F}$ and can be seen as a measure of our prior knowledge on the sources to identify

Main assumption

$\mathbf{F}$ *is a real random vector, whose components are i.i.d. random variables following a Generalized Gaussian distribution*

$$p(\mathbf{F}|\tau_f, q) = \prod_{i=1}^{M} \mathcal{N}_g(F_i|\tau_f, q)$$

Minimal formulation Prior probability distribution

The prior probability distribution reflects the uncertainty related to $\mathbf{F}$ and can be seen as a measure of our prior knowledge on the sources to identify

Main assumption

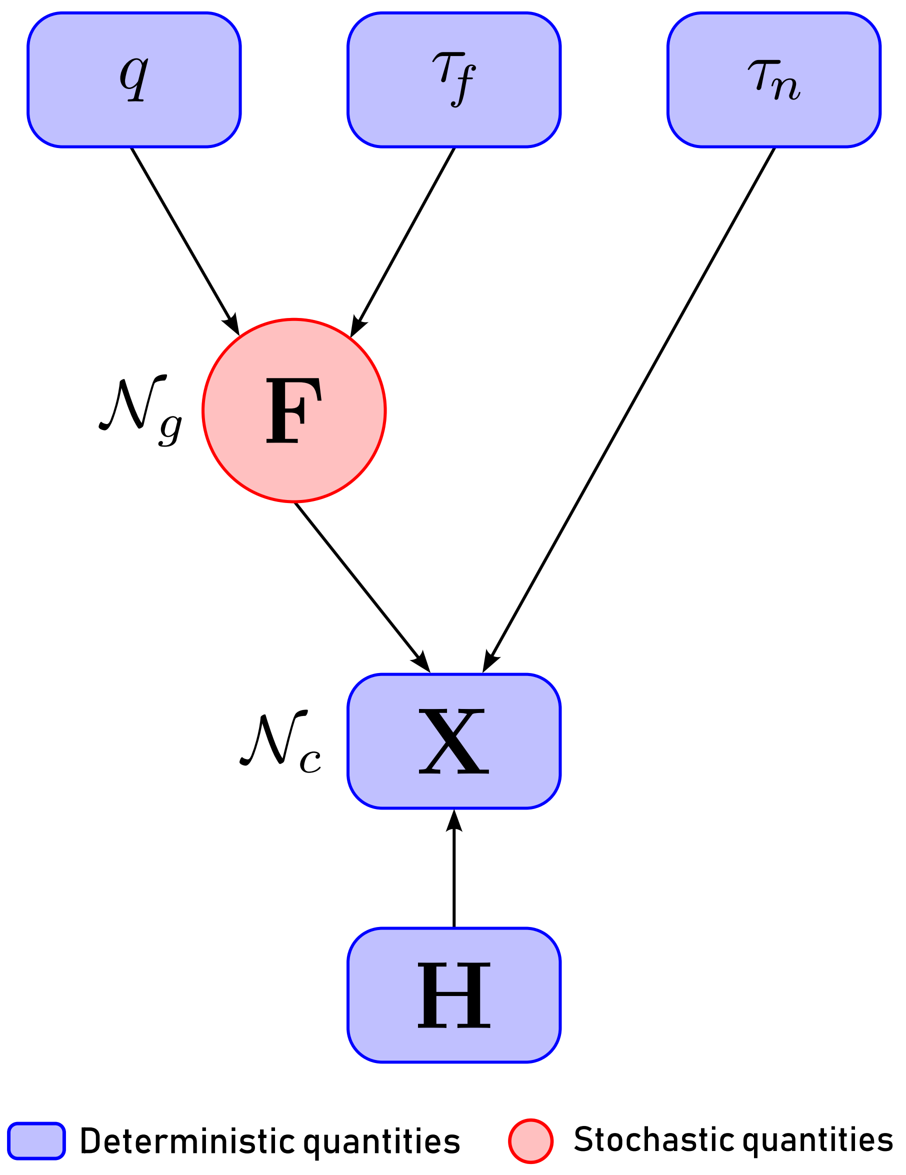
$\mathbf{F}$ is a real random vector, whose components are i.i.d. random variables following a Generalized Gaussian distribution

$$p(\mathbf{F}|\tau_f, q) = \left(\frac{q}{2\Gamma(1/q)}\right)^M \tau_f^{\frac{M}{q}} \exp\left(-\tau_f \|\mathbf{F}\|_q^q\right)$$

- q - Shape parameter of the distribution ($q > 0$)
- τ_f - Scale parameter of the distribution ($\tau_f > 0$)
- $\Gamma(x)$ - Gamma function
- M - Number of reconstruction points

Minimal formulation **Overview**

$$p(\mathbf{F}|\mathbf{X}, \tau_n, \tau_f, q) \propto p(\mathbf{X}|\mathbf{F}, \tau_n)\, p(\mathbf{F}|\tau_f, q)$$



Possible exploitations

- **MAP estimation** – Optimization
- **Uncertainty quantification** – Sampling

Minimal formulation MAP estimation

The MAP estimation consists in finding the most probable excitation field $\mathbf{F}$ given the available data $\mathbf{X}$, the precision parameters (τ_n, τ_f) and the shape parameter q

$$\widehat{\mathbf{F}} = \underset{\mathbf{F}}{\operatorname{argmax}}\ p(\mathbf{F}|\mathbf{X}, \tau_n, \tau_f, q)$$

Minimal formulation MAP estimation

The MAP estimation consists in finding the most probable excitation field $\mathbf{F}$ given the available data $\mathbf{X}$, the precision parameters (τ_n, τ_f) and the shape parameter q

$$\widehat{\mathbf{F}} = \underset{\mathbf{F}}{\text{argmax}}\ p(\mathbf{X}|\mathbf{F}, \tau_n)\, p(\mathbf{F}|\tau_f, q)$$

Minimal formulation MAP estimation

The MAP estimation consists in finding the most probable excitation field $\mathbf{F}$ given the available data $\mathbf{X}$, the precision parameters (τ_n, τ_f) and the shape parameter q

$$\widehat{\mathbf{F}} = \underset{\mathbf{F}}{\text{argmin}} \quad -\log\left[p(\mathbf{X}|\mathbf{F}, \tau_n)\right] - \log\left[p(\mathbf{F}|\tau_f, q)\right]$$

Minimal formulation **MAP estimation**

The MAP estimation consists in finding the most probable excitation field $\mathbf{F}$ given the available data $\mathbf{X}$, the precision parameters (τ_n, τ_f) and the shape parameter q

$$\widehat{\mathbf{F}} = \underset{\mathbf{F}}{\text{argmin}} \;\; \|\mathbf{X} - \mathbf{HF}\|_2^2 + \lambda \|\mathbf{F}\|_q^q \;\; \text{with} \;\; \lambda = \frac{\tau_f}{\tau_n}$$

MAP estimation $\equiv \ell_q$-regularization !

Minimal formulation Uncertainty quantification

Idea for posterior sampling Transform the Generalized Gaussian into a multivariate Gaussian distribution

$$p(\mathbf{F}|\tau_f, q) \propto \exp\left(-\tau_f \|\mathbf{L}\mathbf{F}\|_2^2\right)$$

where $\mathbf{L}^H\mathbf{L} = \mathbf{W}$ is a weigthing depending on $\mathbf{F}$ and q

In doing so, one has

$$\begin{aligned} p(\mathbf{F}|\mathbf{X}) &\propto \exp\left(-\tau_n \|\mathbf{X} - \mathbf{H}\mathbf{F}\|_2^2 - \tau_f \|\mathbf{L}\mathbf{F}\|_2^2\right) \\ &\propto \mathcal{N}_c(\mathbf{F}|\boldsymbol{\mu}_\mathbf{F}, \boldsymbol{\Sigma}_\mathbf{F}) \end{aligned}$$

where $\boldsymbol{\mu}_\mathbf{F} = \tau_n \boldsymbol{\Sigma}_\mathbf{F} \mathbf{H}^H \mathbf{X}$ and $\boldsymbol{\Sigma}_\mathbf{F} = \left(\tau_n \mathbf{H}^H \mathbf{H} + \tau_f \mathbf{W}\right)^{-1}$

Drawing samples

$$\mathbf{F}^{(k)} = \boldsymbol{\mu}_\mathbf{F} + \mathbf{S}\,\mathbf{z}^{(k)} \quad \text{with} \quad \mathbf{S}\mathbf{S}^H = \boldsymbol{\Sigma}_\mathbf{F} \quad \text{and} \quad \mathbf{z}^{(k)} \sim \mathcal{N}_c(\mathbf{z}^{(k)}|\mathbf{0}, \mathbf{I})$$

Properties of Gaussian distributions

Minimal formulation Uncertainty quantification

Estimation of τ_n and τ_f

$\boldsymbol{\mu}_{\mathbf{F}}$ is the solution of the ℓ_q-regularization $\Rightarrow$ After convergence of the iterative process, one obtains $\boldsymbol{\mu}_{\mathbf{F}}$, $\mathbf{W}$ and λ

From these quantities, the most probable values of τ_n and τ_f given the data are computed such that

$$(\widehat{\tau}_n, \widehat{\tau}_f) = \underset{(\tau_n, \tau_f)}{\text{argmax}}\; p(\tau_n, \tau_f | \mathbf{X})$$

Minimal formulation Uncertainty quantification

Estimation of τ_n and τ_f

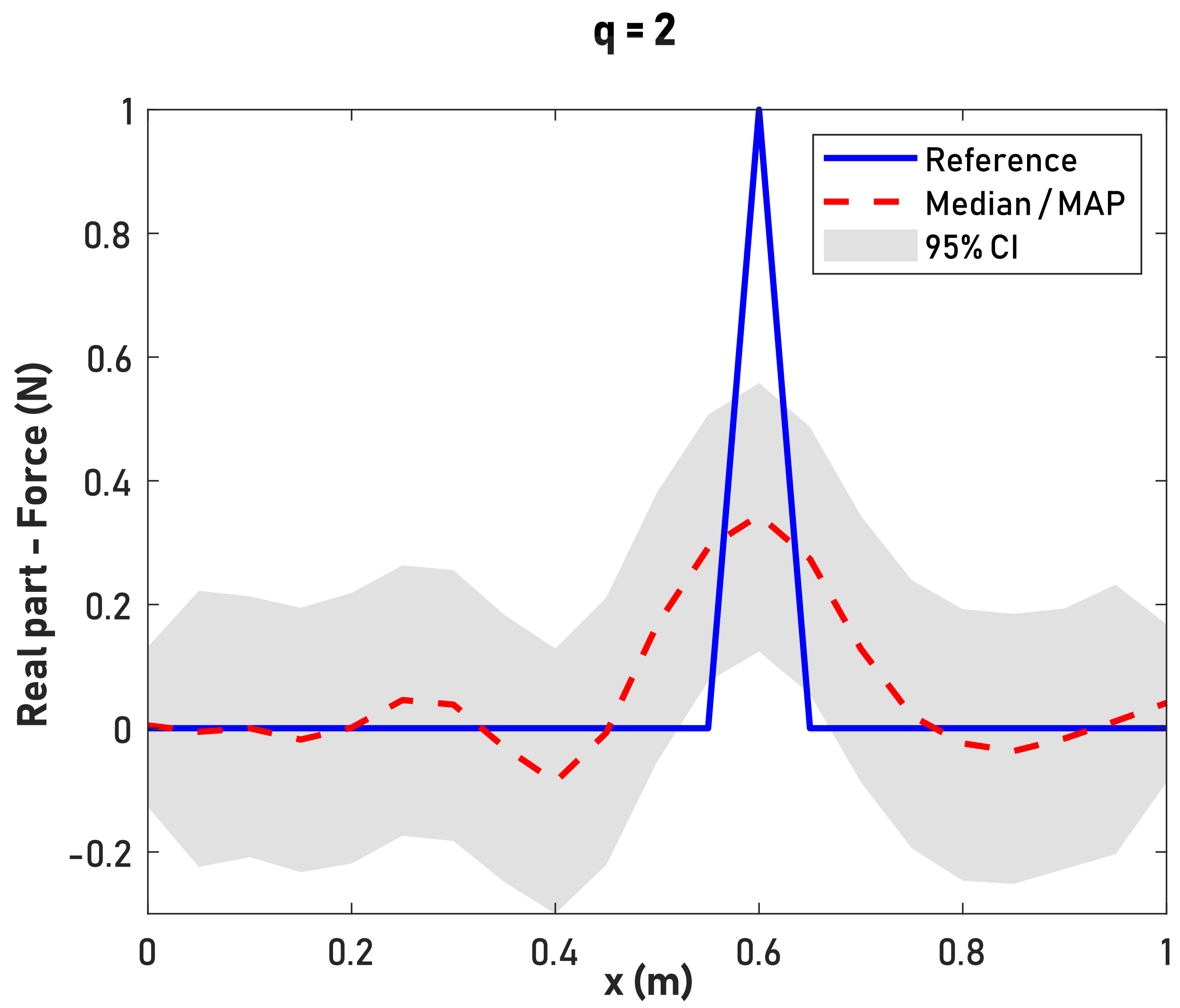
$\boldsymbol{\mu}_{\mathbf{F}}$ is the solution of the ℓ_q-regularization $\Rightarrow$ After convergence of the iterative process, one obtains $\boldsymbol{\mu}_{\mathbf{F}}$, $\mathbf{W}$ and λ

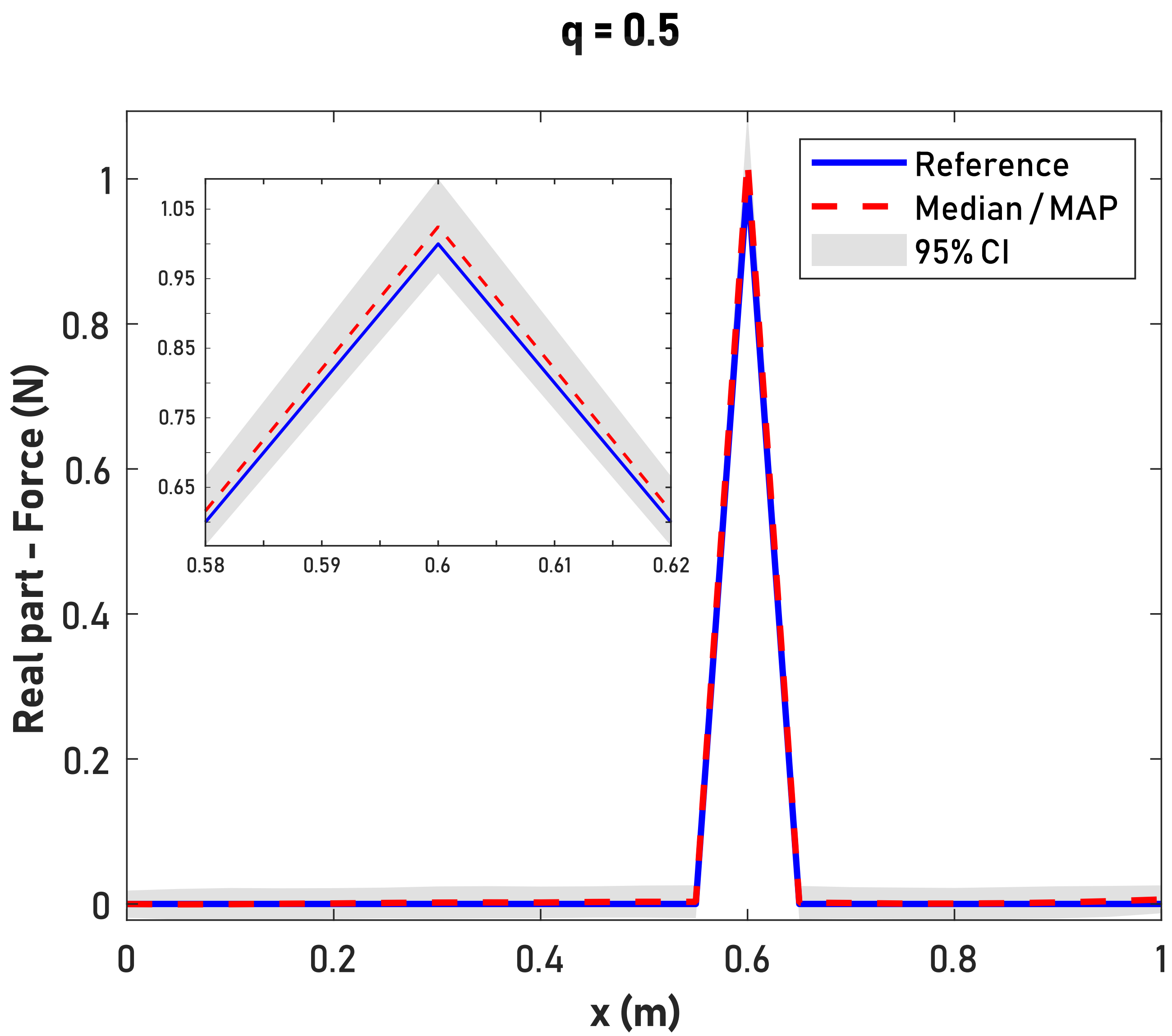
From these quantities, the most probable values of τ_n and τ_f given the data are computed such that

$$\widehat{\tau}_f = \frac{N}{\mathbf{X}^H \left(\mathbf{H}\mathbf{W}^{-1}\mathbf{H}^H + \lambda\mathbf{I}\right)^{-1}\mathbf{X}} \quad \text{and} \quad \widehat{\tau}_n = \frac{\widehat{\tau}_f}{\lambda}$$

Proof

Minimal formulation Application





Minimal formulation Summary

- ✓ MAP is equivalent to ℓ_q-regularization
- ✓ Easy implementation of uncertainty quantification

Provided that...

- ∼ External procedures is implemented to estimate the precision parameters τ_n and τ_f
- ∼ The shape parameter q is known a priori

Need for a more comprehensive formulation

Complete formulation Basics

Choosing a priori relevant values for τ_n, τ_f and q is far from an easy task for non-experienced users $\Rightarrow$ **Infer them !**

Main assumption **τ_n, τ_f and q are independent random variables**

$$p(\mathbf{F}, \tau_n, \tau_f, q|\mathbf{X}) \propto p(\mathbf{X}|\mathbf{F}, \tau_n)\, p(\mathbf{F}|\tau_f, q)\, p(\tau_n)\, p(\tau_f)\, p(q)$$

- $p(\tau)$ - Prior distrubution on the precision parameter τ
- $p(q)$ - Prior distribution on the shape parameter q

How to choose $p(\tau)$ and $p(q)$?

Complete formulation — Prior distribution $p(\tau)$ - Gamma distribution

$$p(\tau|\alpha,\beta) = \mathcal{G}(\tau|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\tau^{\alpha-1}\exp(-\beta\tau) \text{ with } \alpha > 0,\ \beta > 0$$

- α - Scale parameter
- β - Shape parameter

This choice is made for mathematical convenience (conjugate prior), but it does not reflect any real prior information on the precision parameters, except their positiveness

➡ Prior distribution on τ should be as minimally informative as possible (flat prior). For this reason, $\alpha = 1$ and $\beta = 10^{-18}$

Complete formulation **Prior distribution $p(q)$ - Truncated Gamma distribution**

$$p(q|\alpha_q, \beta_q, l_b, u_b) = \frac{\Gamma(\alpha_q)}{\gamma(\alpha_q, \beta_q u_b) - \gamma(\alpha_q, \beta_q l_b)} \mathcal{G}(q|\alpha_q, \beta_q) \mathbb{I}_{[l_b, u_b]}(q)$$

- $\mathbb{I}_{[l_b,u_b]}(q)$ - Truncation function, defined such that

$$\mathbb{I}_{[l_b,u_b]}(q) = \begin{cases} 1 & \text{if } q \in [l_b, u_b] \\ 0 & \text{otherwise} \end{cases}$$

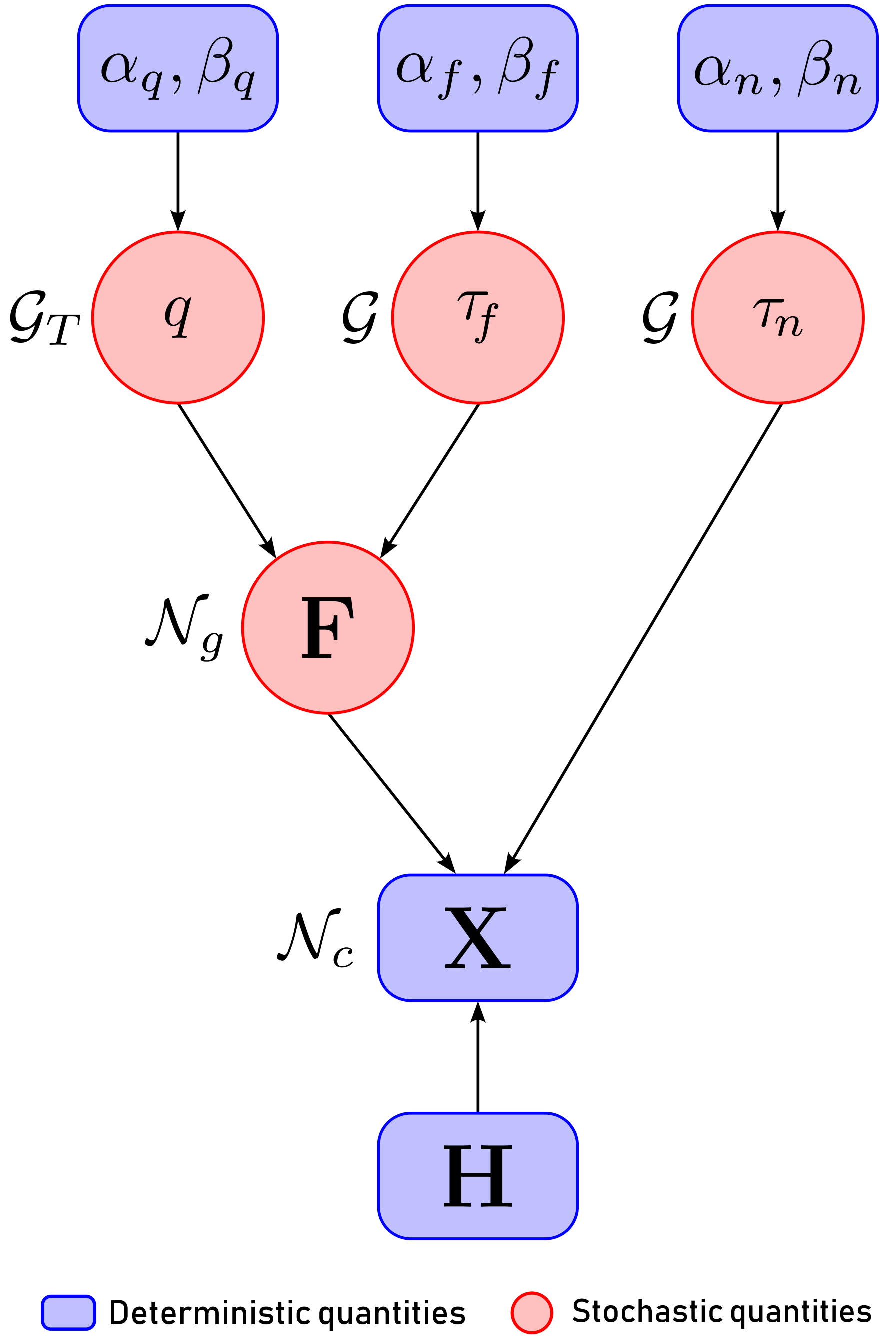
- $\gamma(s, x)$ - Lower incomplete Gamma function

Requirements

- Expert knowledge $\Longrightarrow l_b = 0.05$ and $u_b = 2.05$
- Weakly informative distribution $\Longrightarrow \alpha_q = 1$ and $\beta_q = 10^{-18}$

Complete formulation Overview

$$p(\mathbf{F}, \tau_n, \tau_f, q|\mathbf{X}) \propto$$
$$p(\mathbf{X}|\mathbf{F}, \tau_n)\, p(\mathbf{F}|\tau_f, q)\, p(\tau_n|\alpha_n, \beta_n)\, p(\tau_f|\alpha_f, \beta_f)\, p(q|\alpha_q, \beta_q)$$



Possible exploitations

- **MAP estimation** – Optimization
- **Uncertainty quantification** – Sampling

Complete formulation MAP estimation

The MAP estimate of the complete formulation is given by

$$\left(\widehat{\mathbf{F}}, \widehat{\tau}_n, \widehat{\tau}_n, \hat{q}\right) = \underset{\mathbf{F},\tau_n,\tau_f,q}{\operatorname{argmax}}\, p(\mathbf{F}, \tau_n, \tau_f, q|\mathbf{X})$$

The solution of the previous problem can be found by applying the first-order optimality condition to the dual minimization problem. An alternative, but equivalent, way of solving this problem consists in maximizing the full conditional probability distributions associated to each parameter

$$\begin{aligned}
\hat{q} &= \underset{q}{\operatorname{argmax}}\, p(q|\mathbf{X}, \mathbf{F}, \tau_n, \tau_f) \\
\widehat{\tau}_f &= \underset{\tau_f}{\operatorname{argmax}}\, p(\tau_f|\mathbf{X}, \mathbf{F}, \tau_n, q) \\
\widehat{\tau}_n &= \underset{\tau_n}{\operatorname{argmax}}\, p(\tau_n|\mathbf{X}, \mathbf{F}, \tau_f, q) \\
\widehat{\mathbf{F}} &= \underset{\mathbf{F}}{\operatorname{argmax}}\, p(\mathbf{F}|\mathbf{X}, \tau_n, \tau_f, q)
\end{aligned}$$

Complete formulation MAP estimation

The MAP estimate of the complete formulation is given by

$$\left(\widehat{\mathbf{F}}, \widehat{\tau}_n, \widehat{\tau}_n, \hat{q}\right) = \underset{\mathbf{F},\tau_n,\tau_f,q}{\operatorname{argmax}}\, p(\mathbf{F}, \tau_n, \tau_f, q|\mathbf{X})$$

The solution of the previous problem can be found by applying the first-order optimality condition to the dual minimization problem. An alternative, but equivalent, way of solving this problem consists in maximizing the full conditional probability distributions associated to each parameter

$$\begin{aligned}
\hat{q} &= \underset{q}{\operatorname{argmin}}\ f(q|\widehat{\mathbf{F}}, \widehat{\tau}_f) \\
\widehat{\tau}_f &= \frac{M + \hat{q}(\alpha_f - 1)}{\hat{q}(\beta_f + \|\widehat{\mathbf{F}}\|_{\hat{q}}^{\hat{q}})} \\
\widehat{\tau}_n &= \frac{N + \alpha_n - 1}{\beta_n + \|\mathbf{X} - \mathbf{H}\widehat{\mathbf{F}}\|_2^2} \\
\widehat{\mathbf{F}} &= \underset{\mathbf{F}}{\operatorname{argmin}}\ \|\mathbf{X} - \mathbf{H}\mathbf{F}\|_2^2 + \lambda\|\mathbf{F}\|_{\hat{q}}^{\hat{q}}
\end{aligned}$$

where $f(q|\mathbf{F}, \tau_f) = M \log\Gamma(1/q) - \frac{M}{q}\log\widehat{\tau}_f - [M + \alpha_q - 1]\log q + \beta_q\, q + \widehat{\tau}_f\|\widehat{\mathbf{F}}\|_q^q$ and $\lambda = \widehat{\tau}_f/\widehat{\tau}_n$

Complete formulation MAP estimation – Iterative resolution

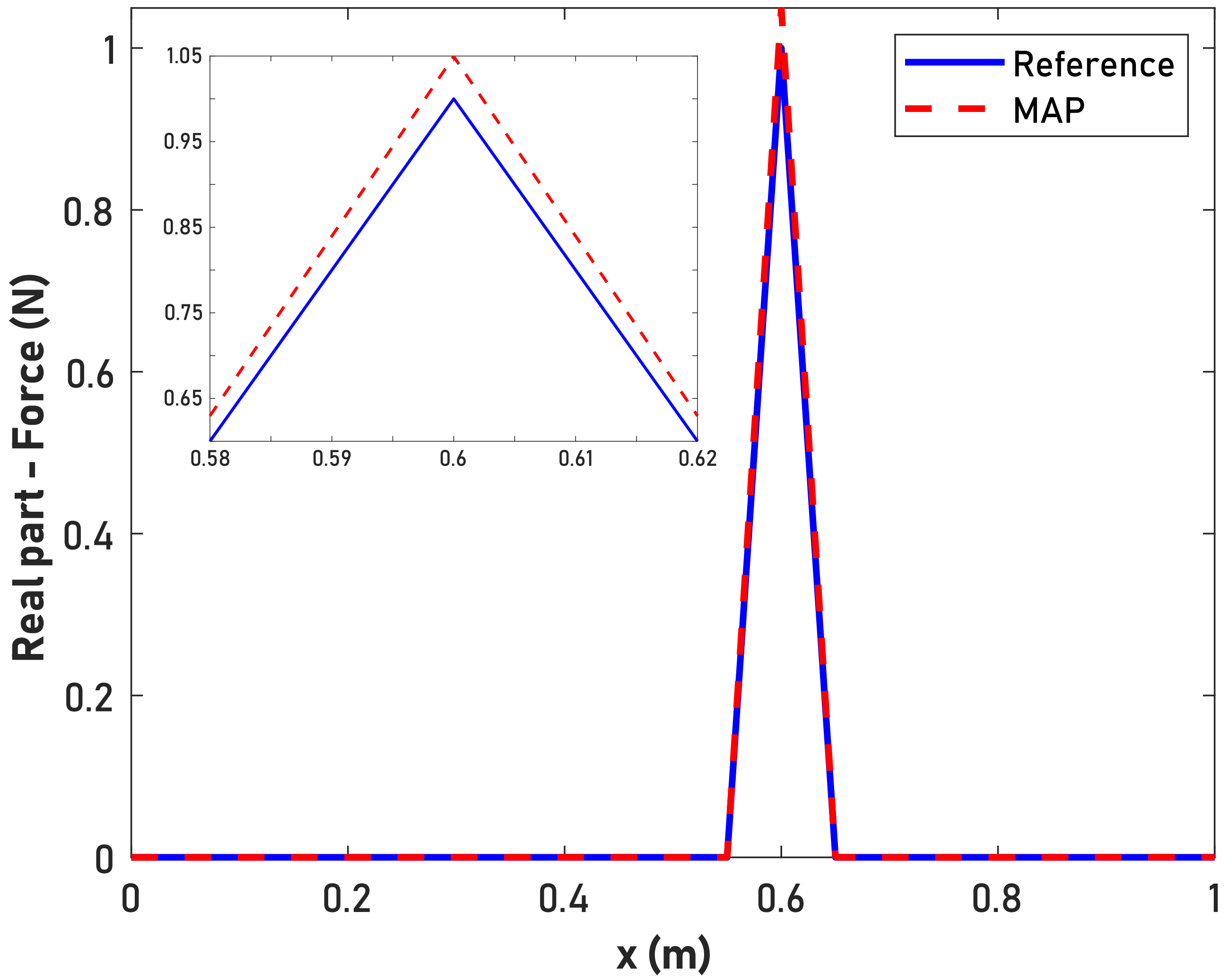
Initialization ℓ_2-regularization $\left(\widehat{\mathbf{F}}^{(0)}, \boldsymbol{\lambda}^{(0)}, \widehat{\mathbf{q}}^{(\mathbf{0})} = \mathbf{2}\right)$ + Estimation of $\boldsymbol{\tau}_{\boldsymbol{f}}^{(\mathbf{0})}$ from $\boldsymbol{\lambda}^{(\mathbf{0})}$

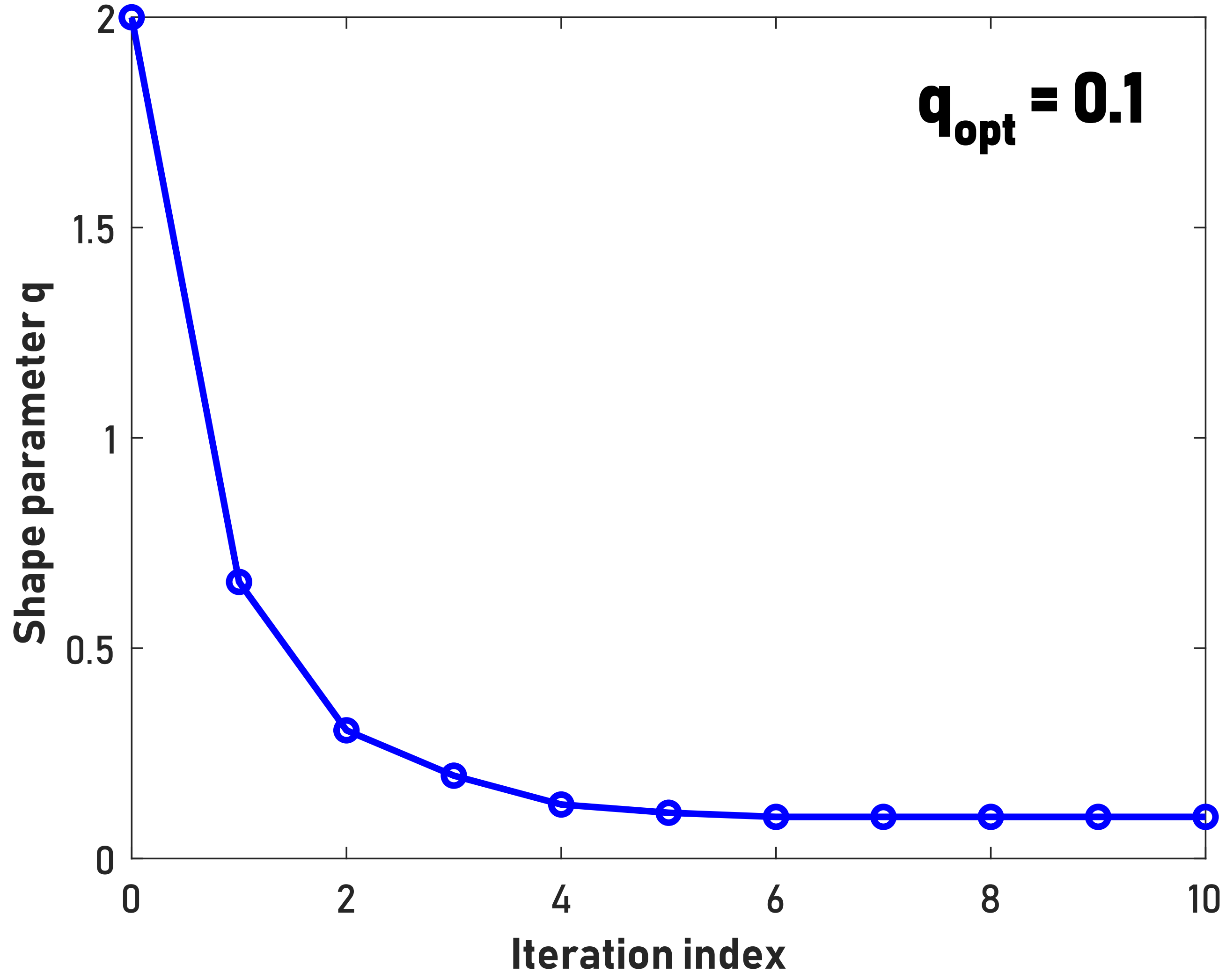
Iteration While convergence is not reached do

$$
\begin{aligned}
\hat{q}^{(k)} &= \underset{q}{\operatorname{argmin}}\ f\left(q|\widehat{\mathbf{F}}^{(k-1)}, \widehat{\tau}_f^{(k-1)}\right) \\
\widehat{\tau}_f^{(k)} &= \frac{M + \hat{q}^{(k)}(\alpha_f - 1)}{\hat{q}^{(k)}\left(\beta_f + \|\widehat{\mathbf{F}}^{(k-1)}\|_{\hat{q}^{(k)}}^{\hat{q}^{(k)}}\right)} \\
\widehat{\tau}_n^{(k)} &= \frac{N + \alpha_n - 1}{\beta_n + \|\mathbf{X} - \mathbf{H}\widehat{\mathbf{F}}^{(k-1)}\|_2^2} \\
\widehat{\mathbf{F}}^{(k)} &= \underset{\mathbf{F}}{\operatorname{argmin}}\ \|\mathbf{X} - \mathbf{H}\mathbf{F}\|_2^2 + \lambda^{(k)}\|\mathbf{F}\|_{\hat{q}^{(k)}}^{\hat{q}^{(k)}}
\end{aligned}
$$

Convergence monitoring $\boldsymbol{\delta} = \|\widehat{\mathbf{F}}^{(\mathbf{k})} - \widehat{\mathbf{F}}^{(\mathbf{k-1})}\|_{\mathbf{1}} / \|\widehat{\mathbf{F}}^{(\mathbf{k-1})}\|_{\mathbf{1}}$

Complete formulation MAP estimation – Application





Complete formulation Uncertainty quantification – MCMC

Markov Chain Monte Carlo (MCMC) is a class of algorithms that produce sequences of random samples converging to a target distribution for which direct sampling is difficult

Here, because the full conditional probability distributions are available, a Gibbs sampler (particular case of MH sampler) can be implemented

$$
\begin{aligned}
p(q|\mathbf{X},\mathbf{F},\tau_n,\tau_f) &\propto \frac{\tau_f^{M/q}}{\Gamma(1/q)} q^{M+\alpha_q-1} \exp\left(-\beta_q\, q - \tau_f \|\mathbf{F}\|_q^q\right) \mathbb{I}_{[l_b,u_b]} \\
p(\tau_f|\mathbf{X},\mathbf{F},\tau_n,q) &\propto \mathcal{G}(\tau_f|\alpha_f + M/q, \beta_f + \|\mathbf{F}\|_q^q) \\
p(\tau_n|\mathbf{X},\mathbf{F},\tau_f,q) &\propto \mathcal{G}(\tau_n|\alpha_n + N, \beta_n + \|\mathbf{X}-\mathbf{HF}\|_2^2) \\
p(\mathbf{F}|\mathbf{X},\tau_n,\tau_f,q) &\propto \exp\left(-\tau_n \|\mathbf{X}-\mathbf{HF}\|_2^2 - \tau_f \|\mathbf{F}\|_q^q\right)
\end{aligned}
$$

Build a markov chain
for each parameter to compute statistics

Complete formulation Uncertainty quantification – Gibbs sampling

General scheme

1. Set $k = 0$ and initialize $q^{(0)}$, $\tau_n^{(0)}$, $\tau_f^{(0)}$ and $\mathbf{F}^{(0)}$
2. Draw N_s samples from full conditional distributions

   for $k = 1 : N_s$
   - Draw $q^{(k)} \sim p\left(q|\mathbf{X}, \mathbf{F}^{(k-1)}, \tau_n^{(k-1)}, \tau_f^{(k-1)}\right)$
   - Draw $\tau_f^{(k)} \sim p\left(\tau_f|\mathbf{X}, \mathbf{F}^{(k-1)}, \tau_n^{(k-1)}, q^{(k)}\right)$
   - Draw $\tau_n^{(k)} \sim p\left(\tau_n|\mathbf{X}, \mathbf{F}^{(k-1)}, \tau_f^{(k)}, q^{(k)}\right)$
   - Draw $\mathbf{F}^{(k)} \sim p\left(\mathbf{F}|\mathbf{X}, \tau_n^{(k)}, \tau_f^{(k)}, q^{(k)}\right)$

   end for
3. Monitor the convergence (stationarity) of the Markov chains

Complete formulation Uncertainty quantification – Implementation

Initialization

- ℓ_2-regularization $\left(\mathbf{F}^{(0)}, \lambda^{(0)}, q^{(0)}\right)$ + Estimation of $\tau_n^{(0)}$ and $\tau_f^{(0)}$ from $\lambda^{(0)}$
- MAP estimate $\left(\mathbf{F}^{(0)}, \tau_f^{(0)}, \tau_n^{(0)}, q^{(0)}\right)$

Drawing samples

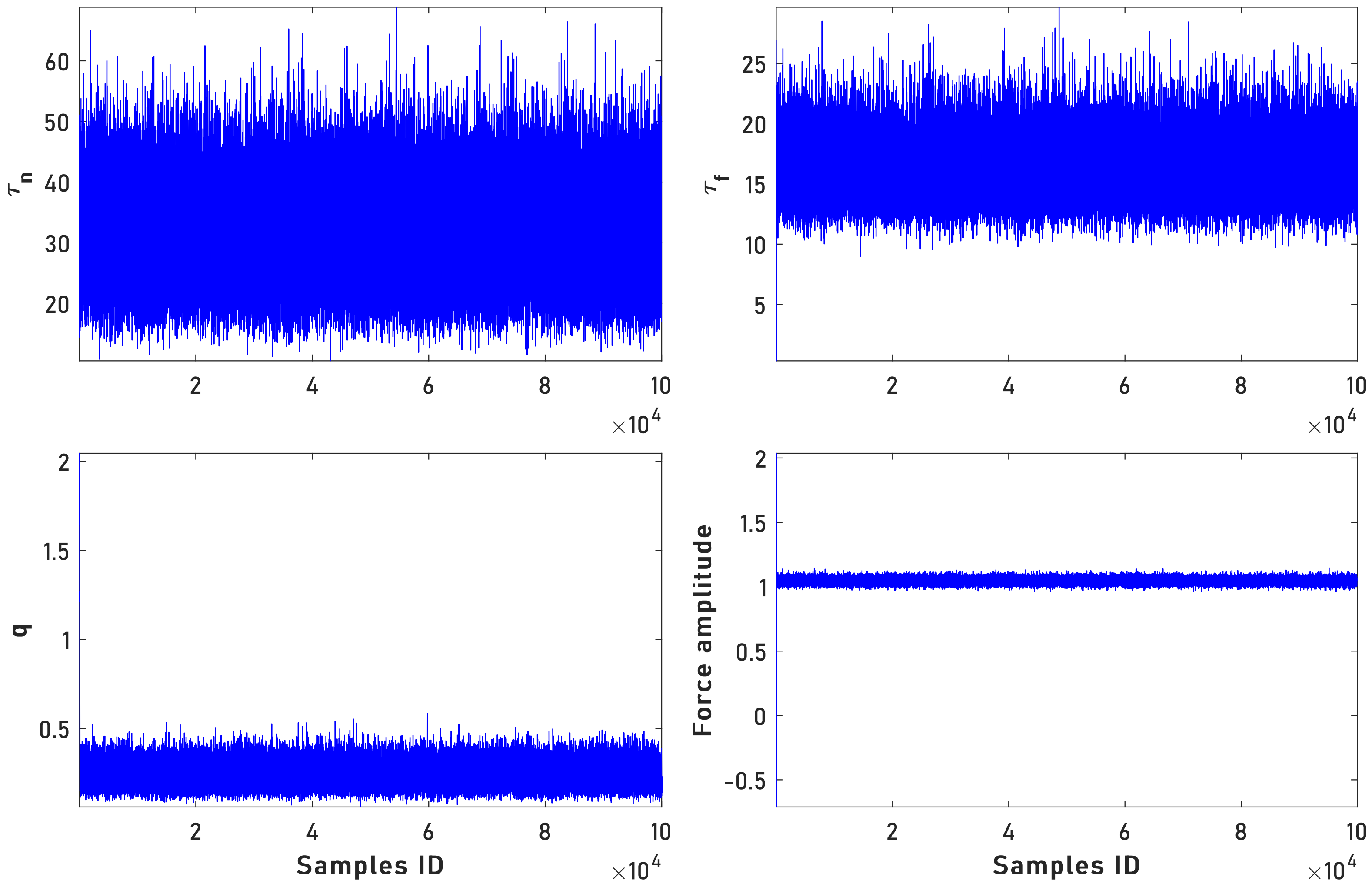
1. $p\left(q | \mathbf{X}, \mathbf{F}^{(k-1)}, \tau_n^{(k-1)}, \tau_f^{(k-1)}\right)$ - Non-standard probability distribution ⇒ Instance of MH sampler (or HMC, ...)
2. $p\left(\tau_i | \mathbf{X}, \mathbf{F}^{(k-1)}, \tau_j^{(k-1)}, q^{(k)}\right)$ - Gamma distribution ⇒ RNG implemented in standard programming languages
3. $p\left(\mathbf{F} | \mathbf{X}, \tau_n^{(k)}, \tau_f^{(k)}, q^{(k)}\right)$ - Multivariate Gaussian-like distribution ⇒ Procedure defined for the min. formulation

Convergence diagnostic

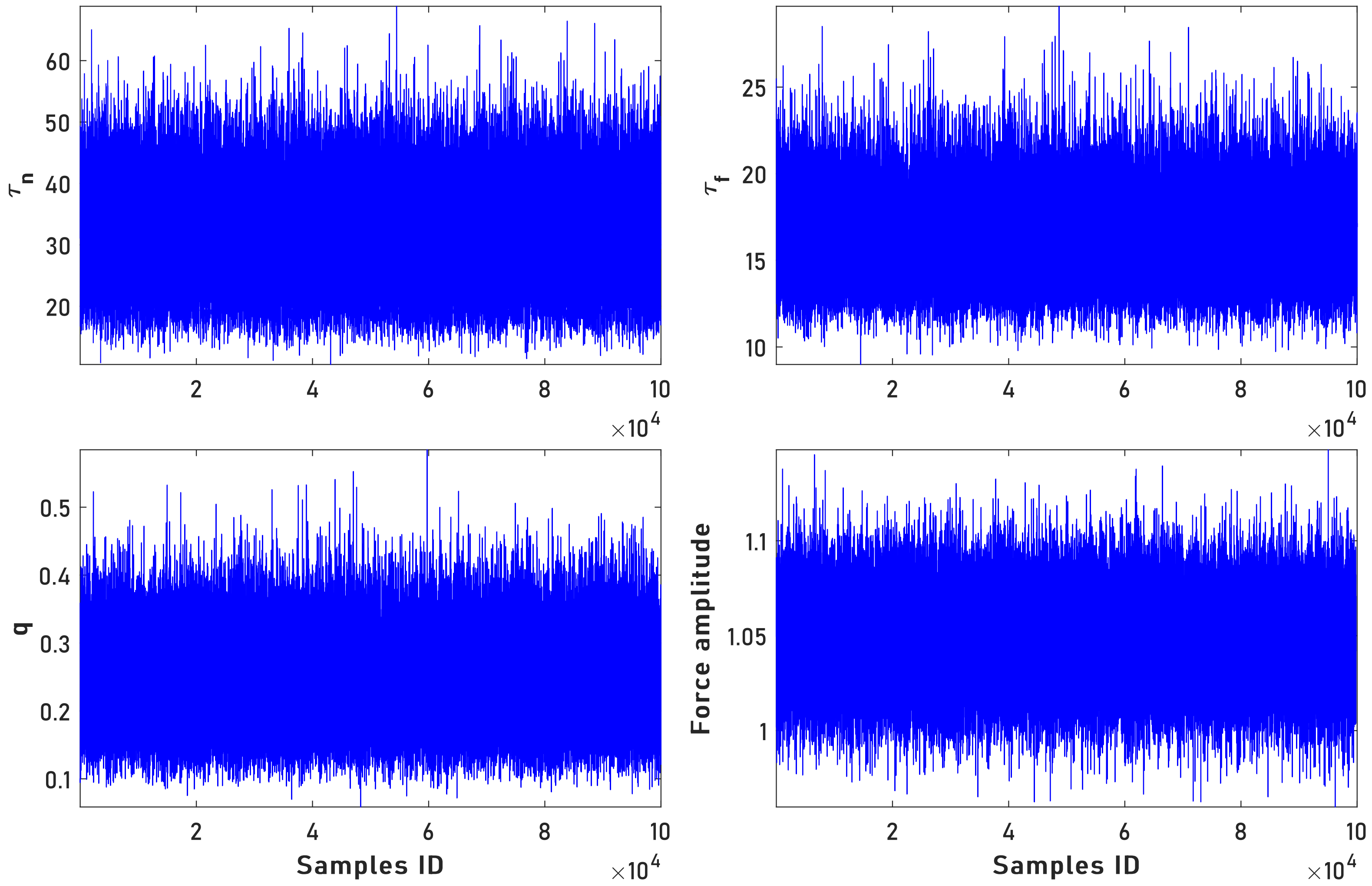
- **Burn-in period** – Number of samples to discard at the beginning of the chains (period before convergence)
- **Total length** – Number of samples required to compute statistics
- **Available diagnostics** – Raftery-Lewis, Geweke (one long chain), Gelman-Rubin (multiple chains) and more

Complete formulation Uncertainty quantification – Application

Initialization : ℓ_2-regularization

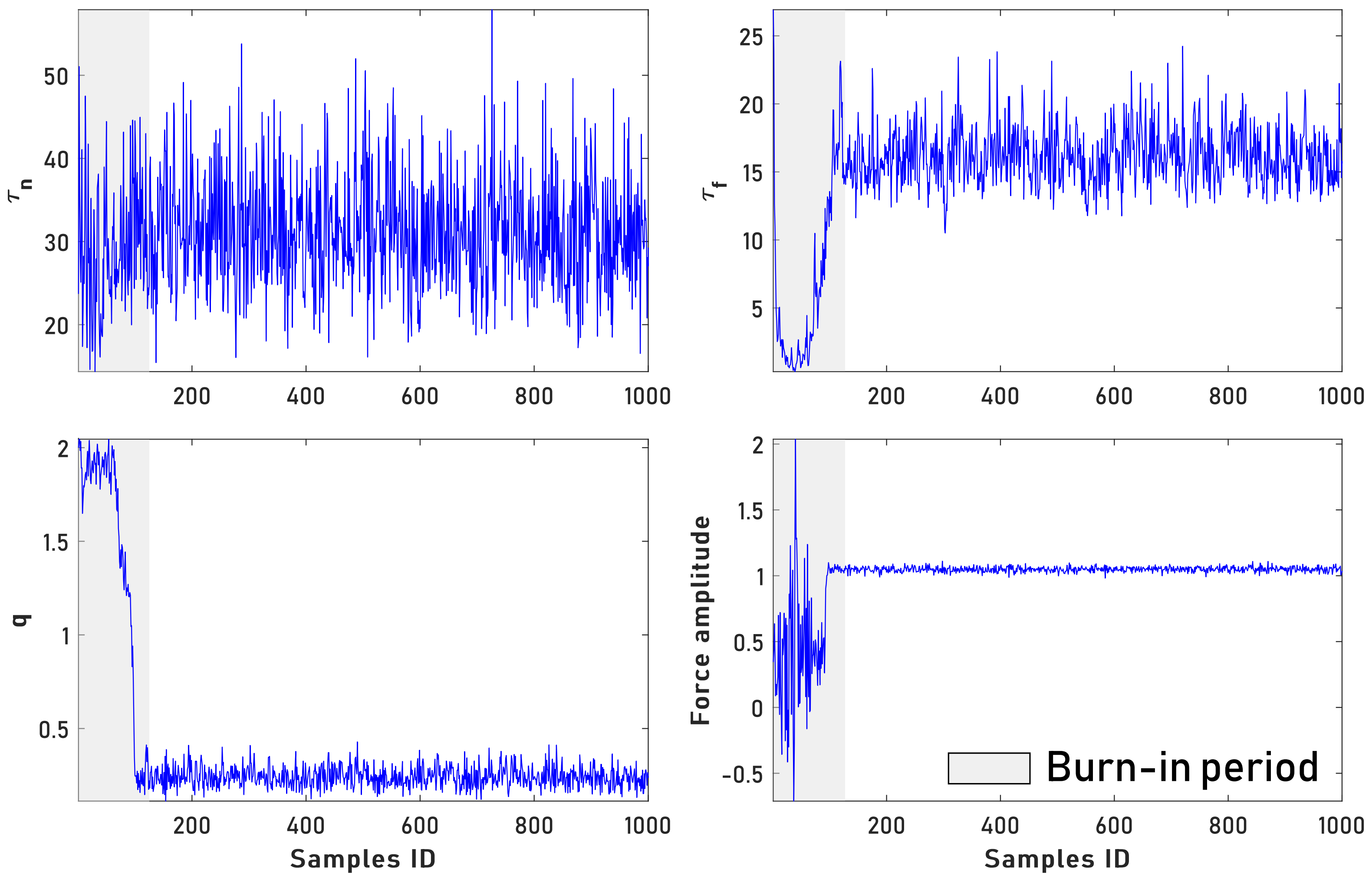


Initialization : MAP estimation

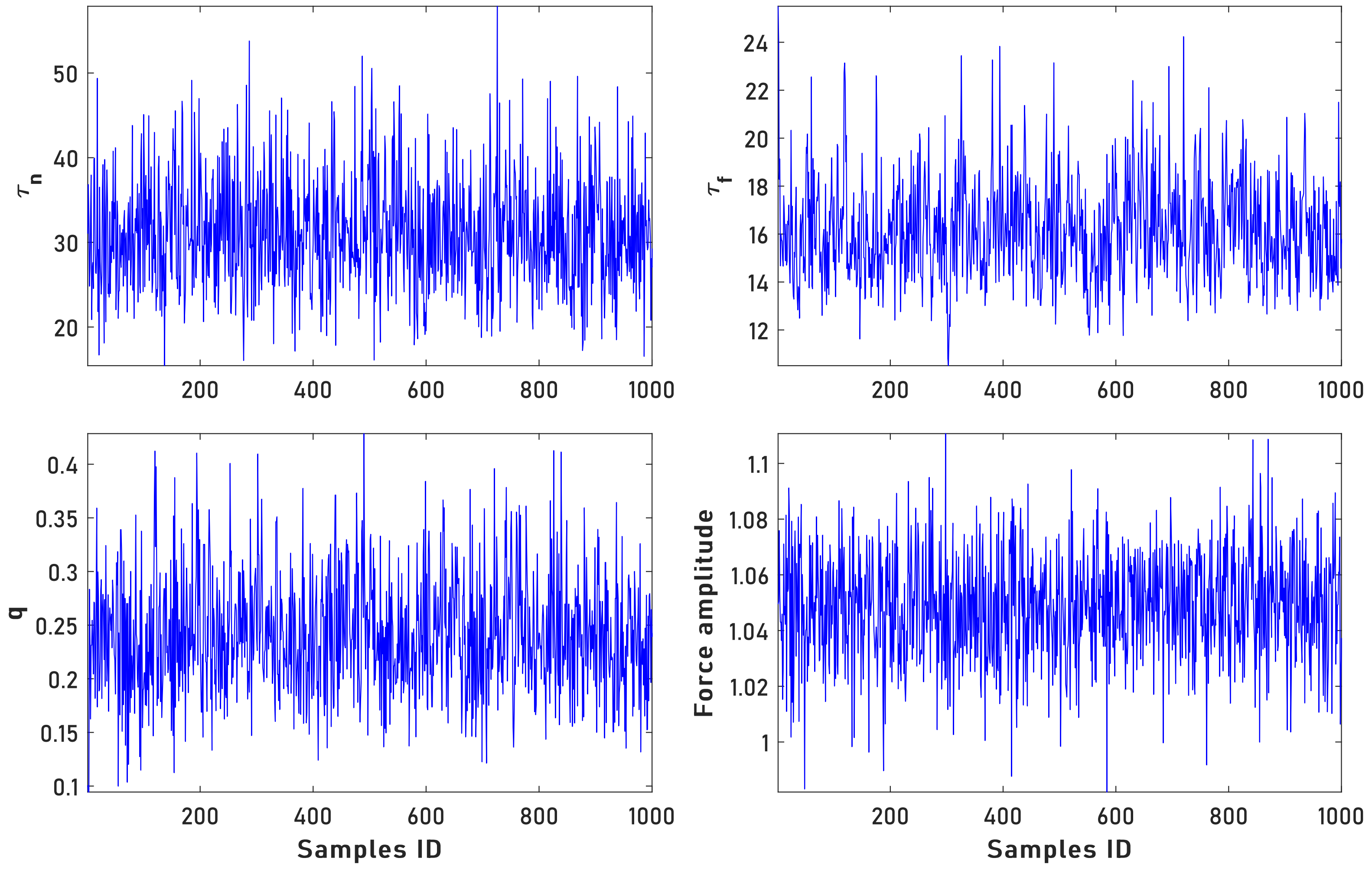


Complete formulation Uncertainty quantification – Application

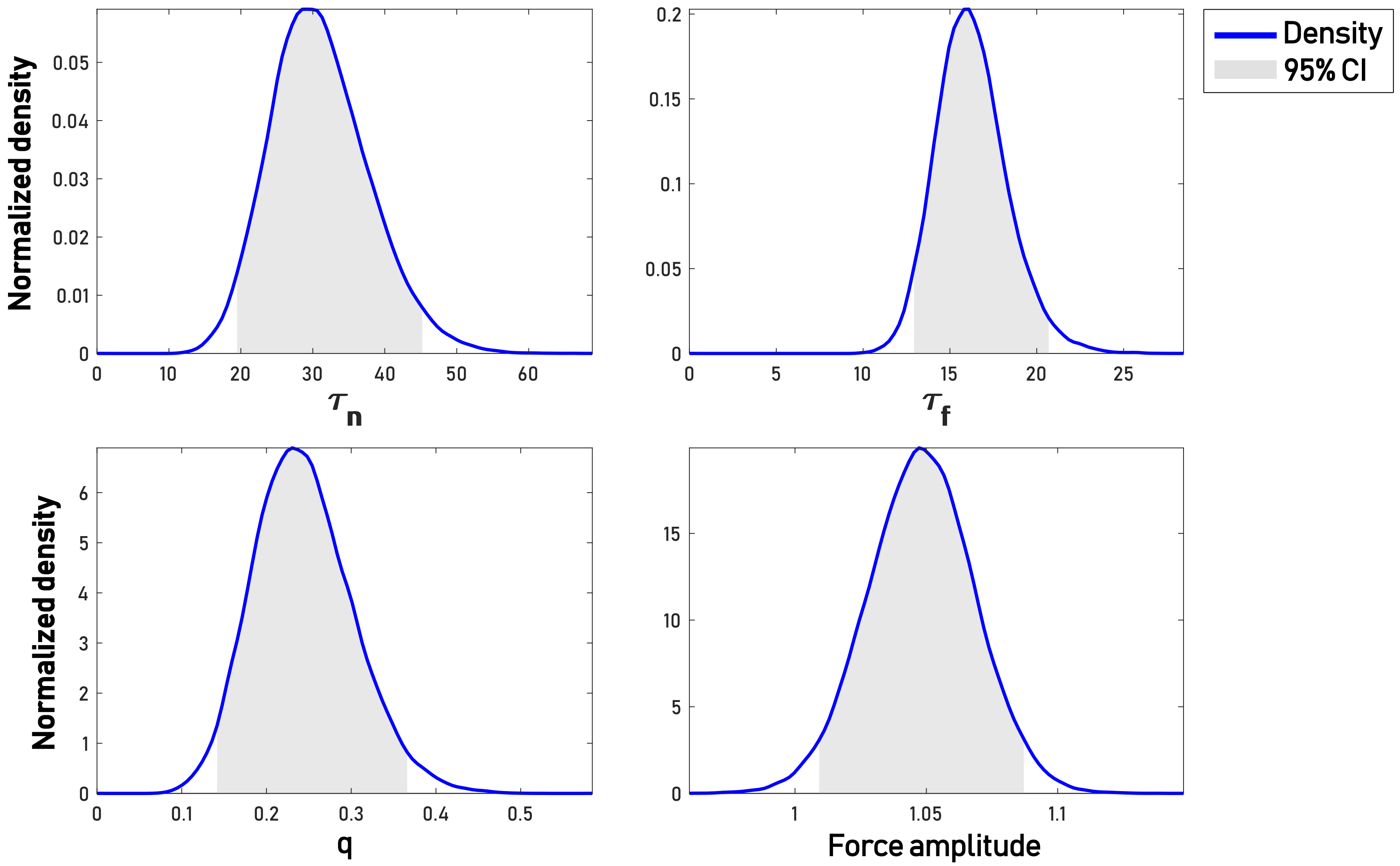
Initialization : ℓ_2-regularization



Initialization : MAP estimation

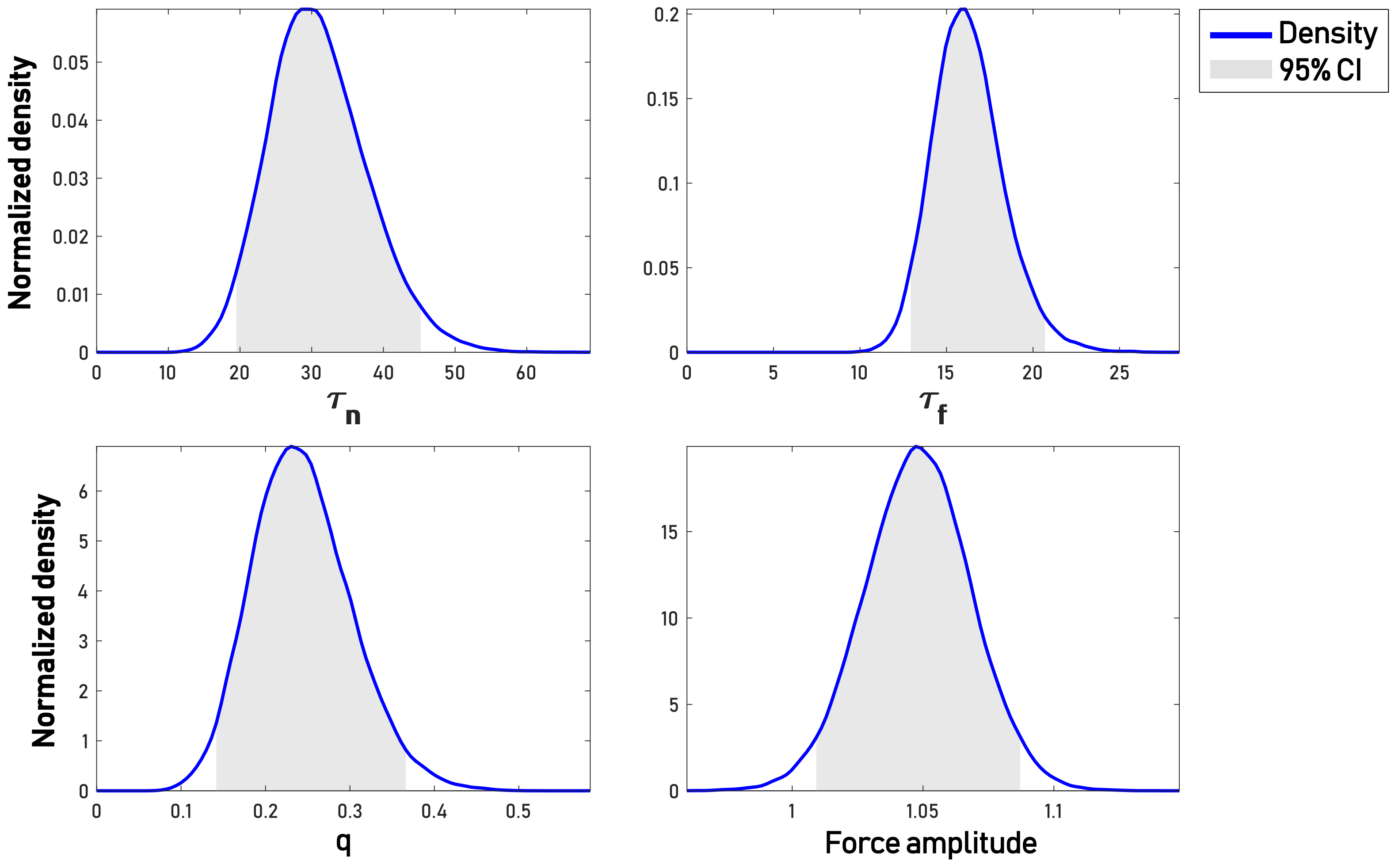


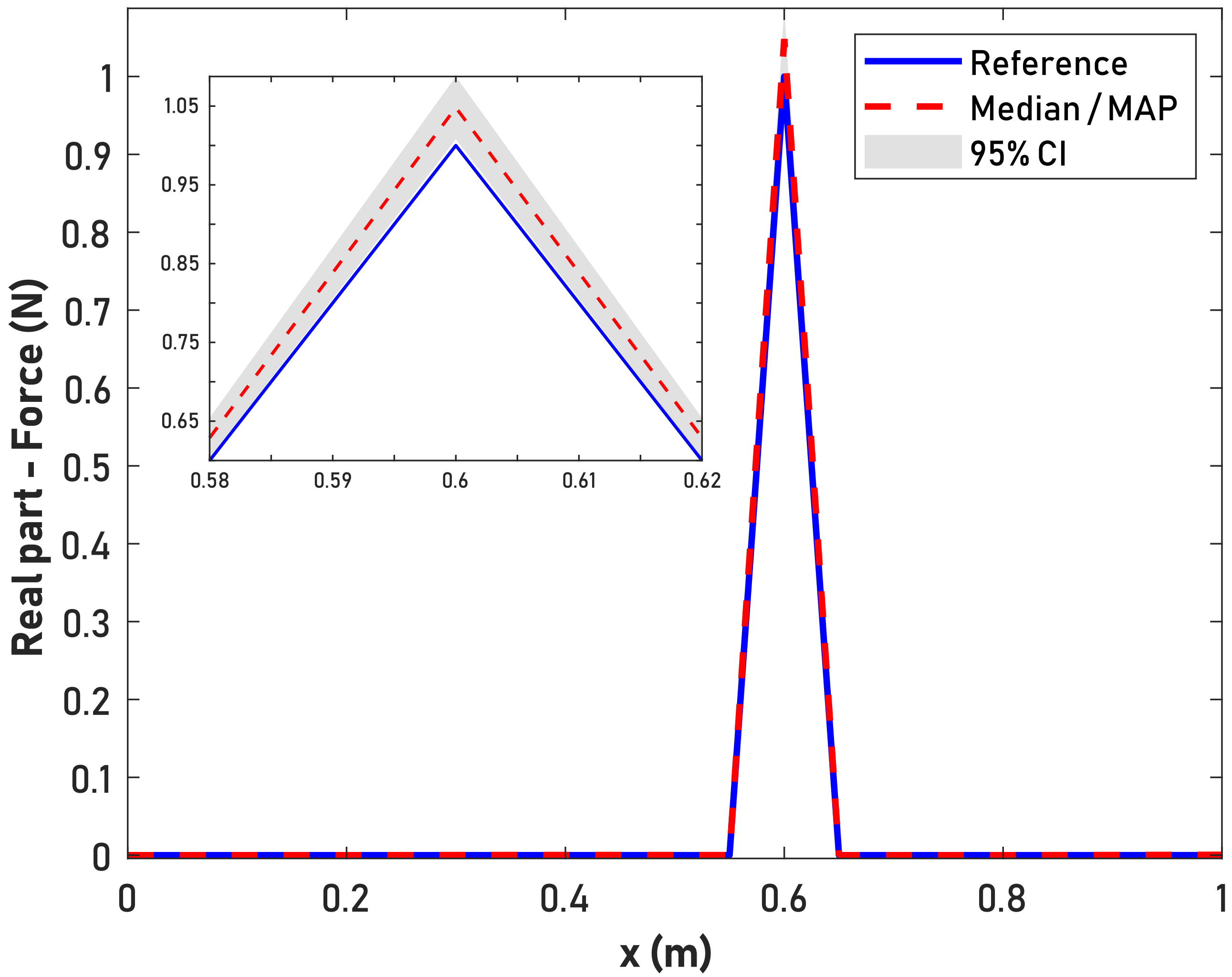
Complete formulation Uncertainty quantification – Application



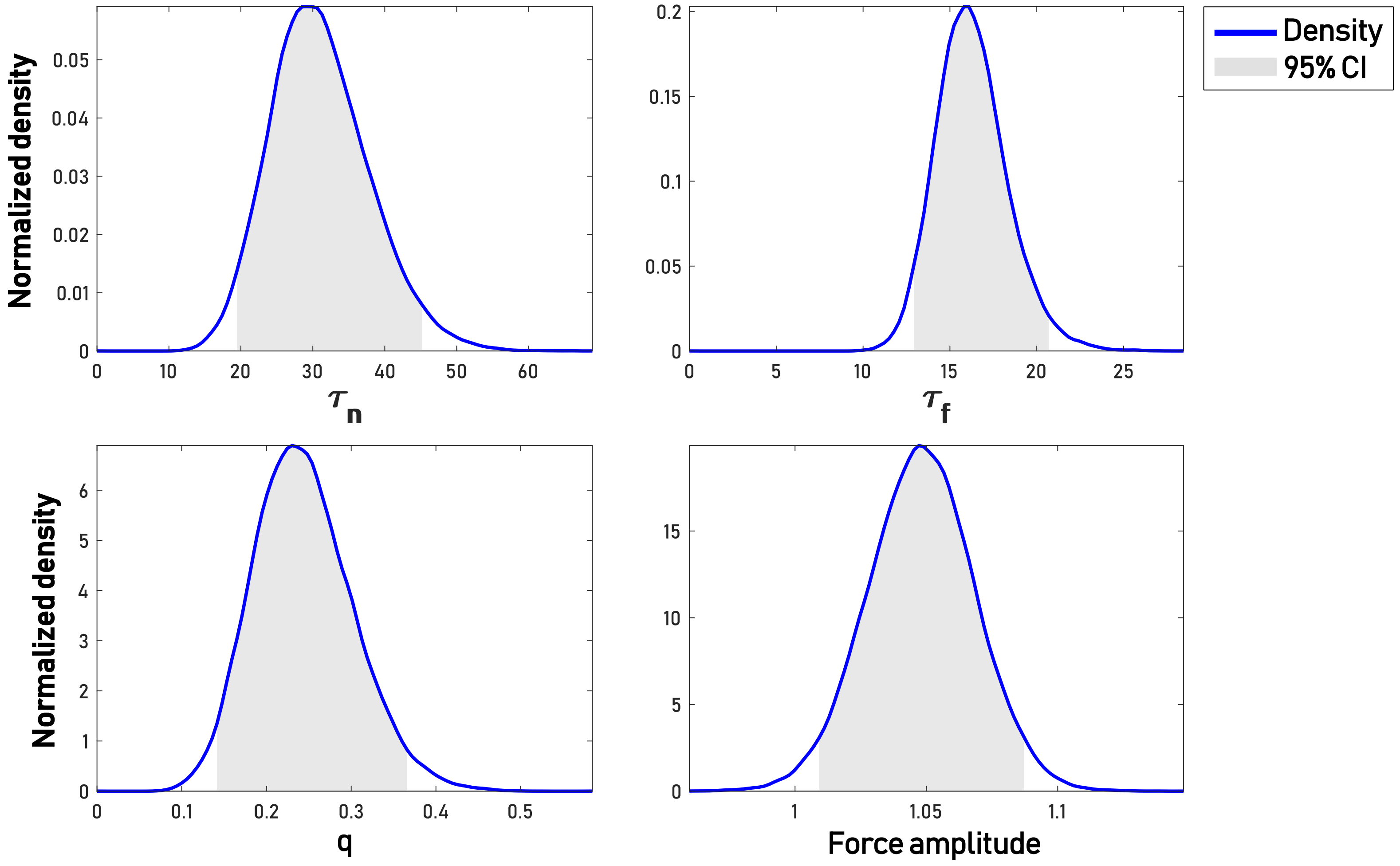
	F_0	τ_n	τ_f	q
Median	1.0481	30.50	16.12	0.240
Mean	1.0480	31.02	16.27	0.244
MAP	1.0472	29.21	16.09	0.230
95% CI	[1.0079, 1.0876]	[19.08, 45.77]	[12.66, 20.76]	[0.141, 0.368]

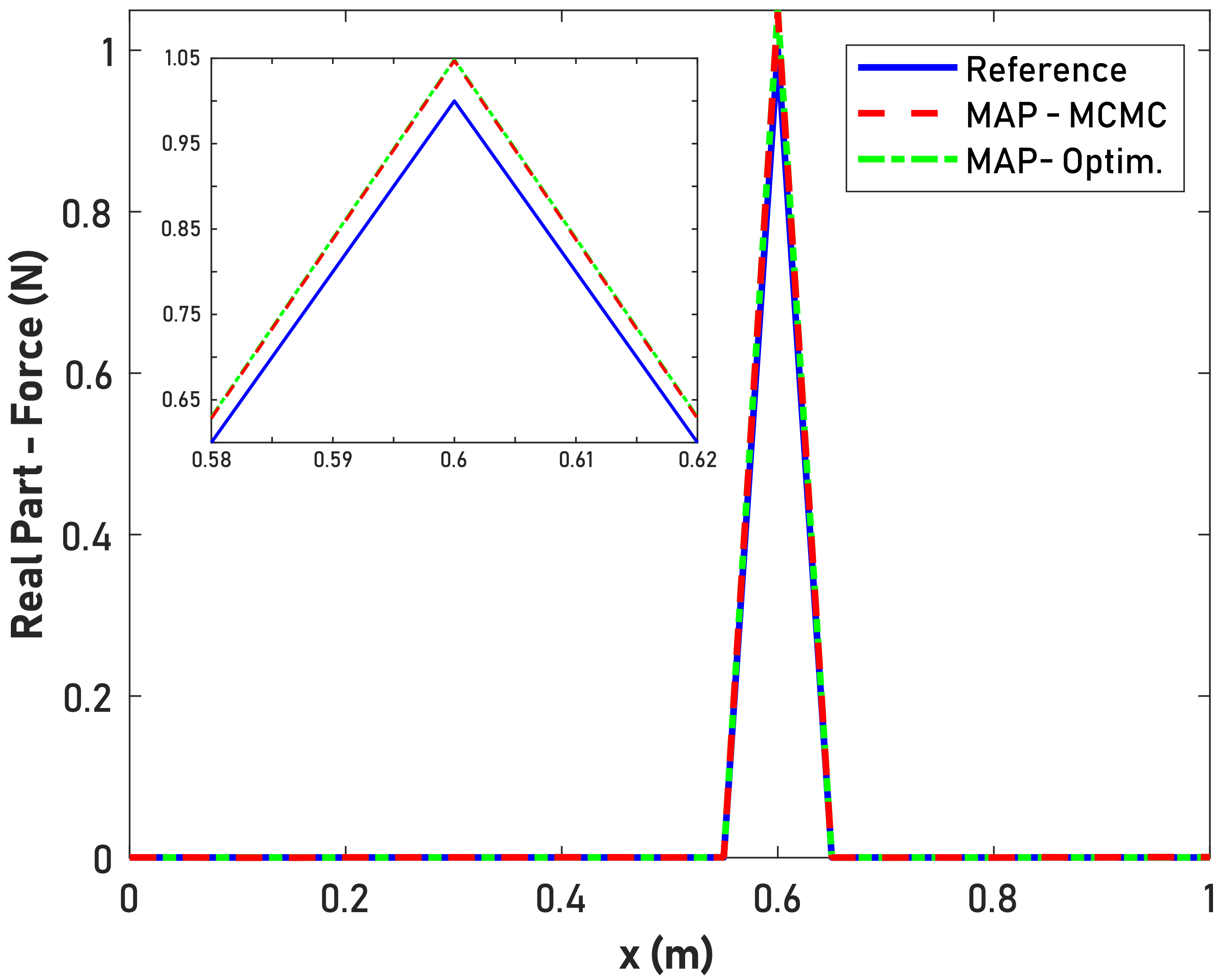
Complete formulation Uncertainty quantification – Application





Complete formulation Uncertainty quantification – Application





Complete formulation **Summary**

- ✓ Automatic identification of all the parameters
- ✓ Robust identification of the excitation field

Can we do better or at least different ?

Yes, of course !

Outline

Relevant Vector Regression Basics

RVR is a particular Bayesian approach for which the prior probability distribution over $\mathbf{F}$ is such that

$$p(\mathbf{F}) = \prod_{i=1}^{M} \mathcal{N}\left(F_i|0, \tau_{fi}^{-1}\right) \text{ with } \mathcal{N}(F_i|0, \tau_{fi}^{-1}) = \sqrt{\frac{\tau_{fi}}{2\pi}} \exp\left(-\frac{\tau_{fi}}{2}|F_i|^2\right)$$

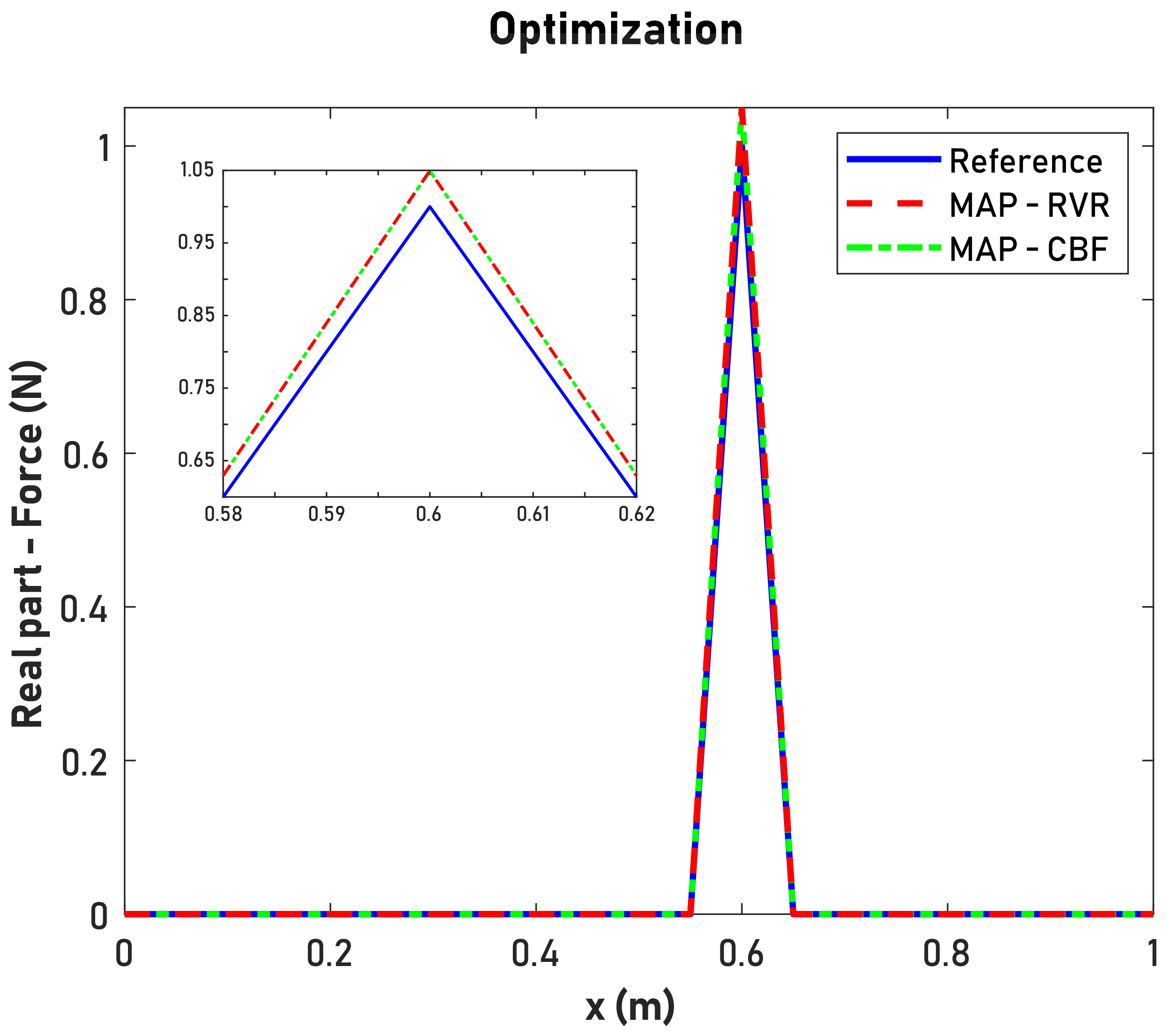
The corresponding Bayesian formulation is expressed as

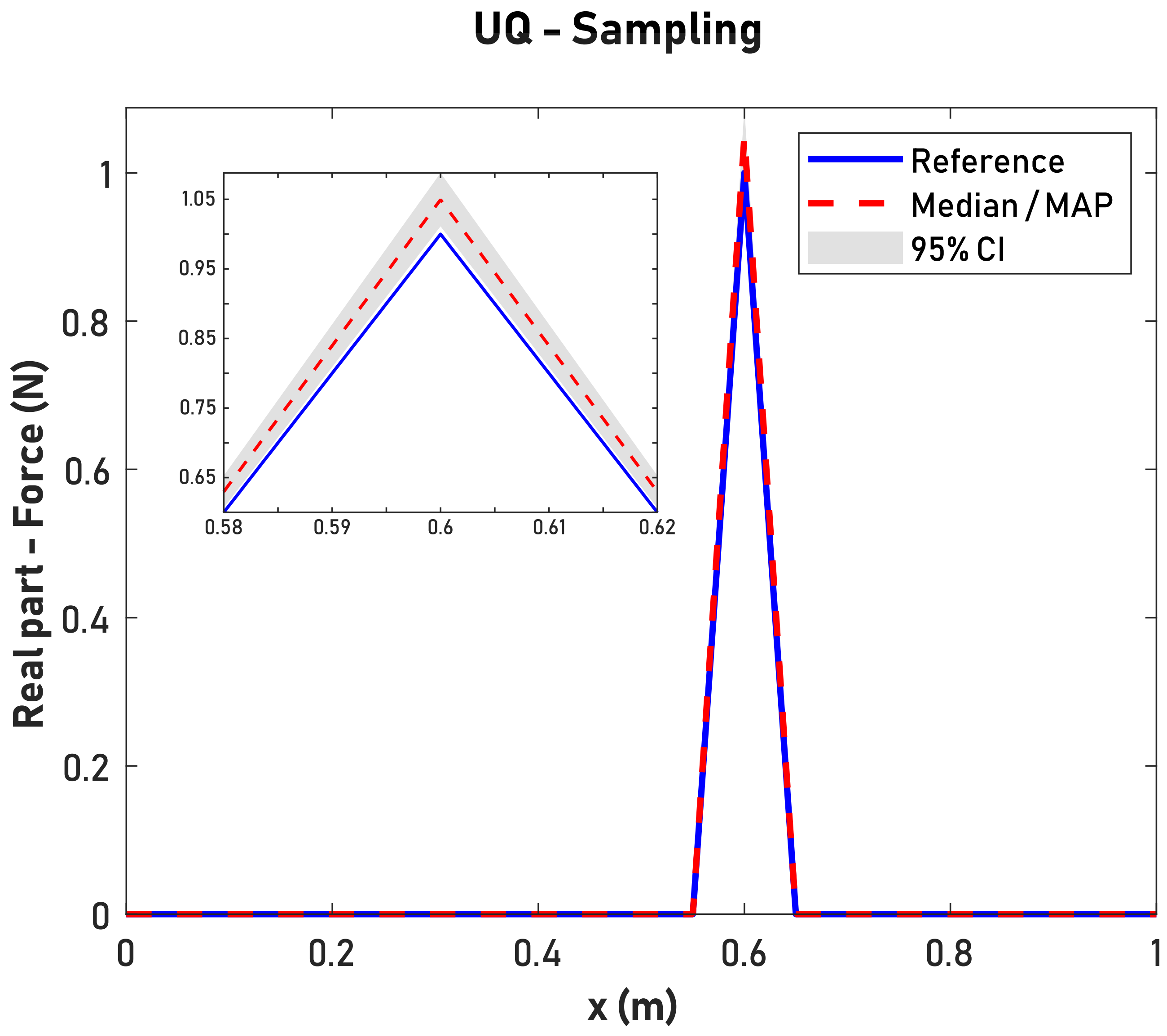
$$p(\mathbf{F}, \tau_n, \tau_{f_i}|\mathbf{X}) \propto p(\mathbf{X}|\mathbf{F}, \tau_n) \prod_{i=1}^{M} p(F_i|\tau_{f_i})\, p(\tau_{fi}) \text{ with } p(\tau_{fi}) = \mathcal{G}(\tau_{f_i}|\alpha_{fi}, \beta_{fi})$$

Main features

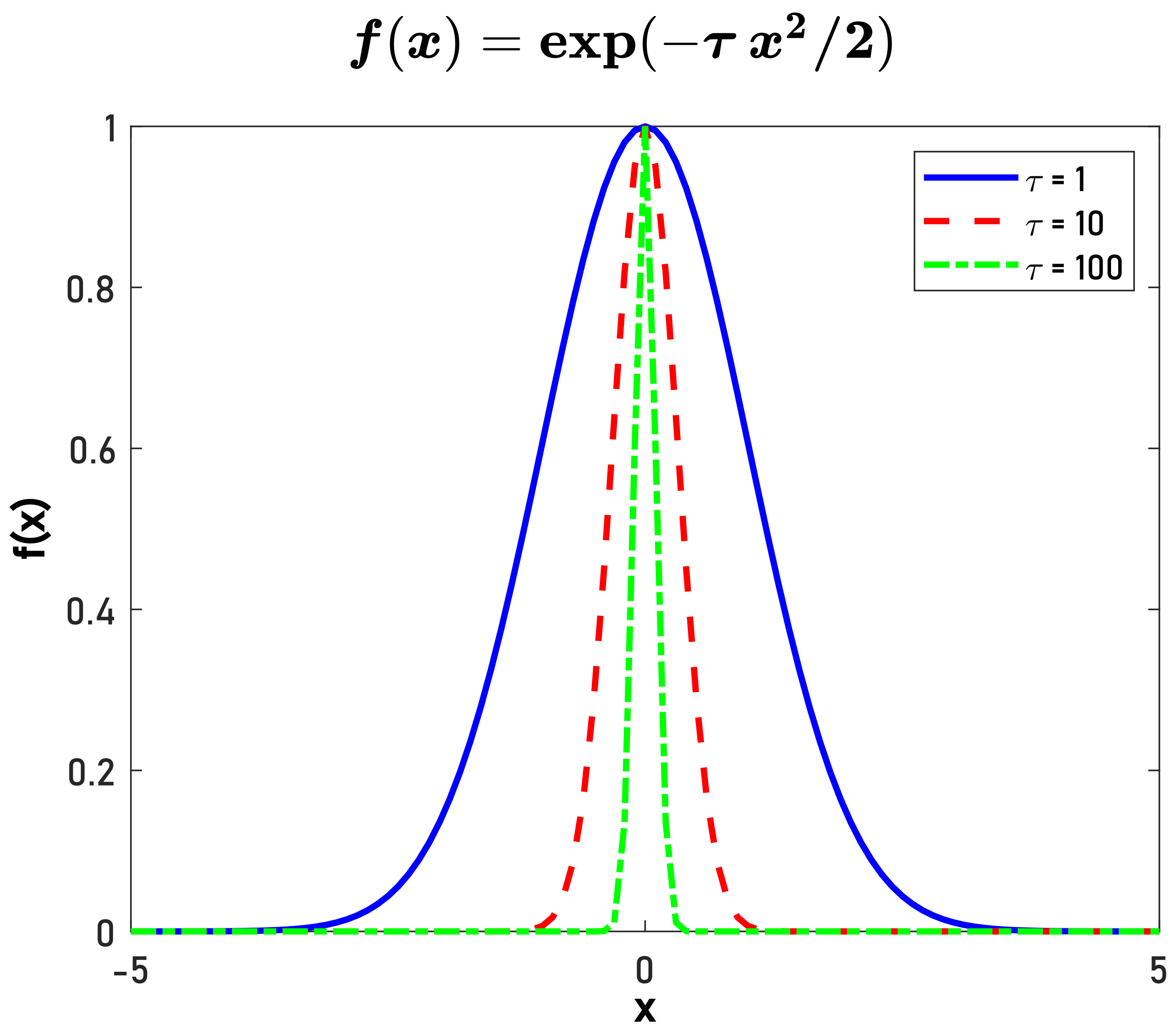
- Implementation of MAP estimation and UQ via Gibbs sampling require minor changes of the algorithms described previously
- More parameters needs to be infered ($M + 3$ for CBF and $2M + 1$ for RVR)
- Computationally more efficient than CBF

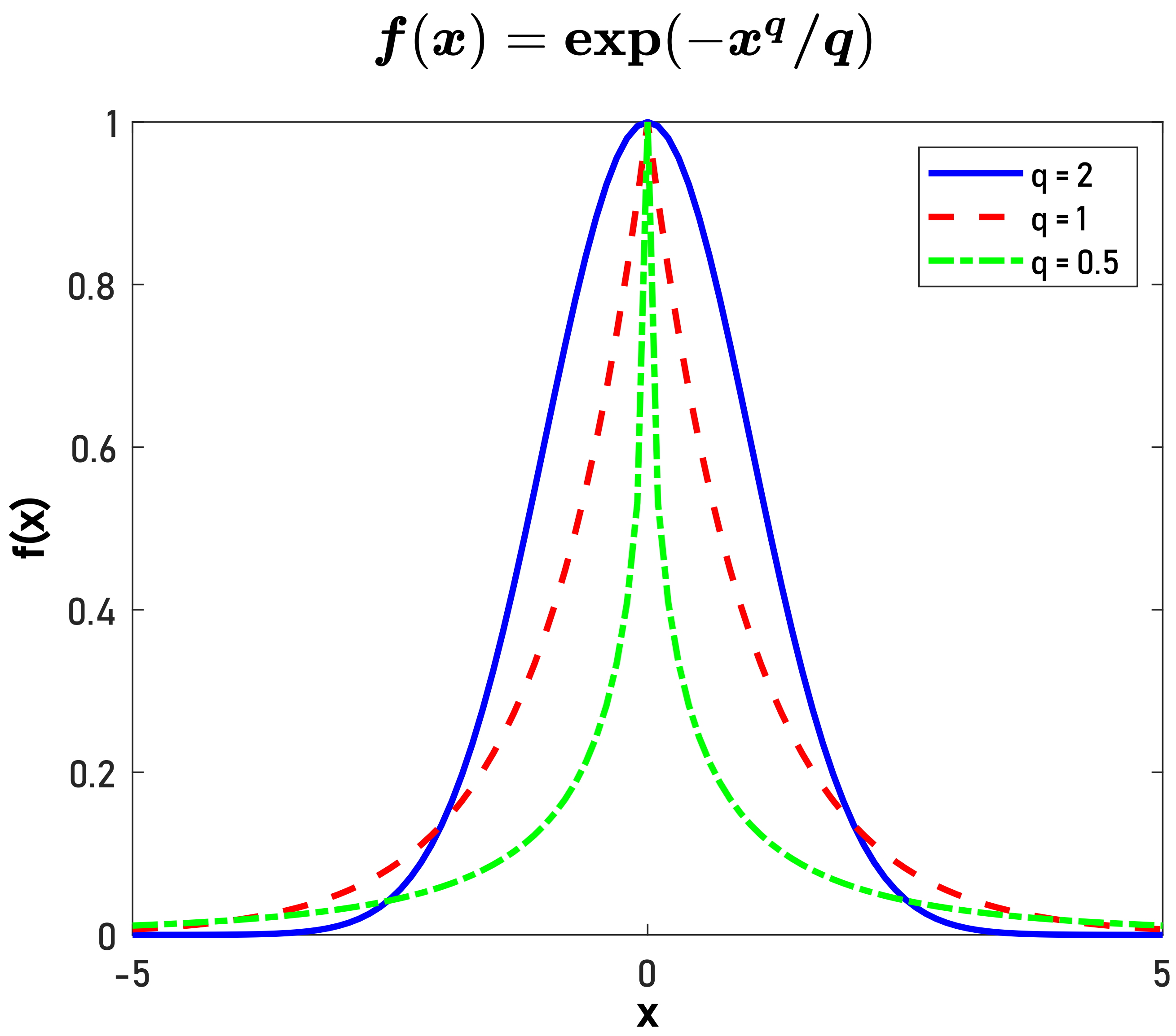
Relevant Vector Regression Application





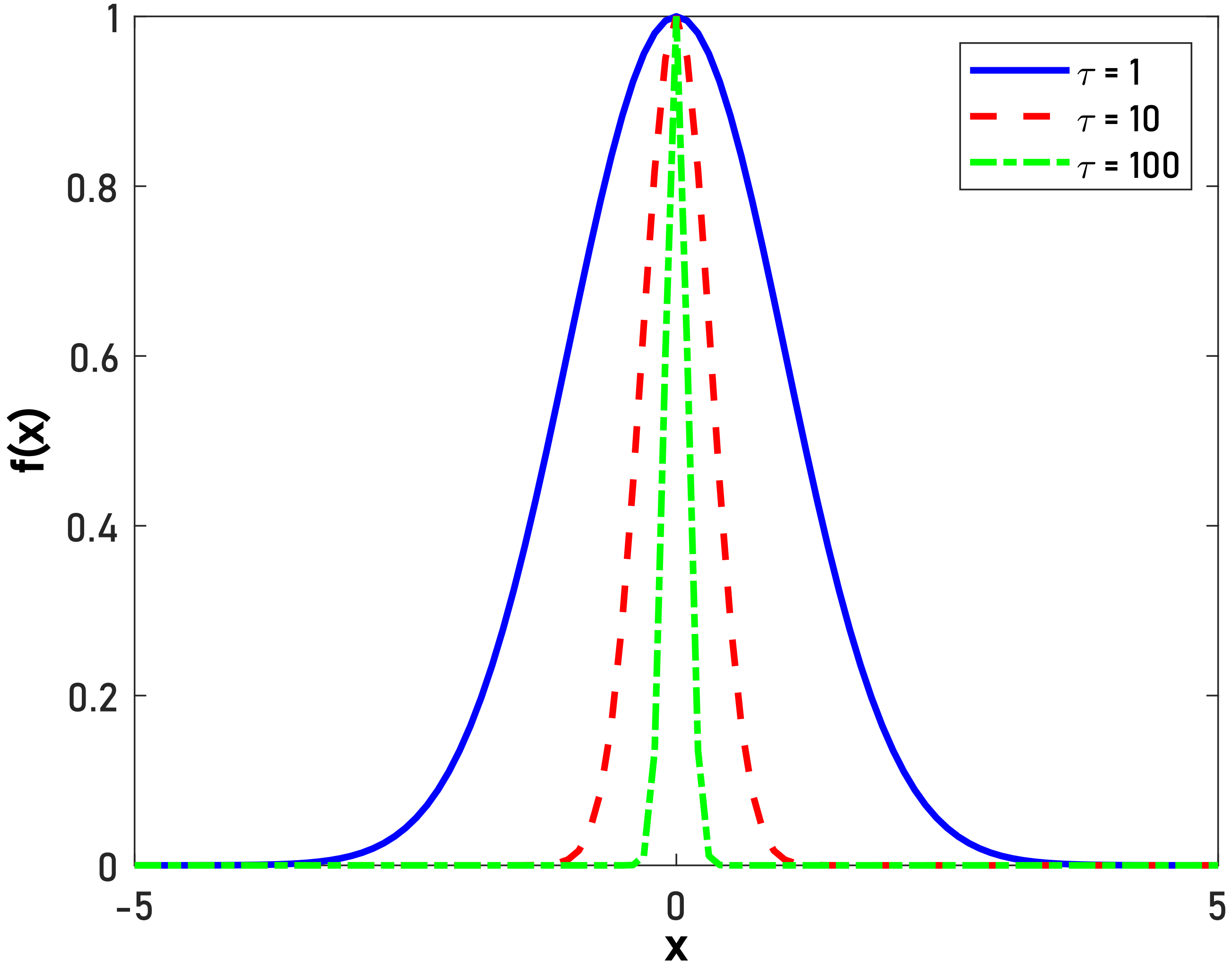
Relevant Vector Regression Why does it work so well ?





Relevant Vector Regression Why does it work so well ?

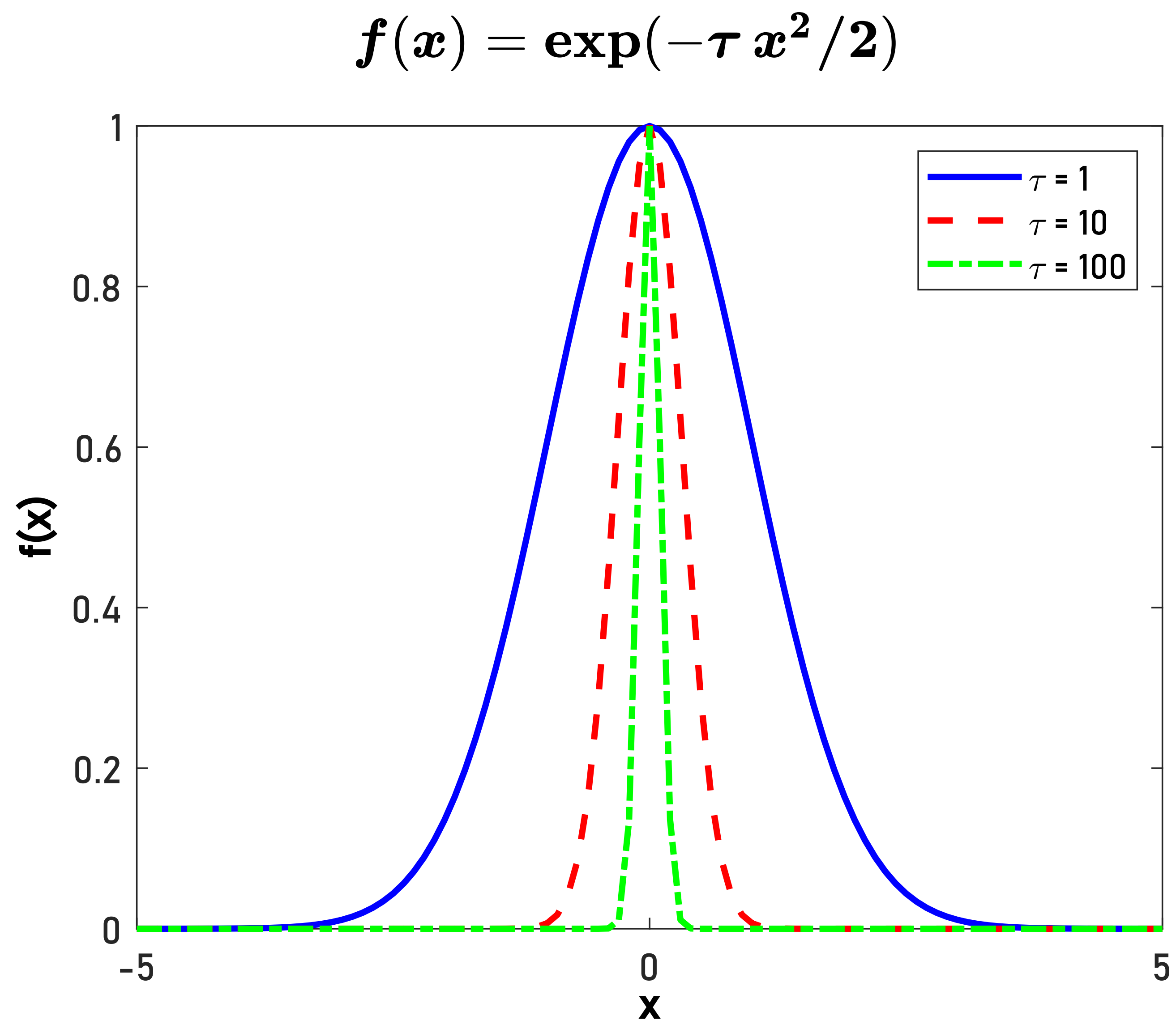
$$f(x) = \exp(-\tau x^2/2)$$

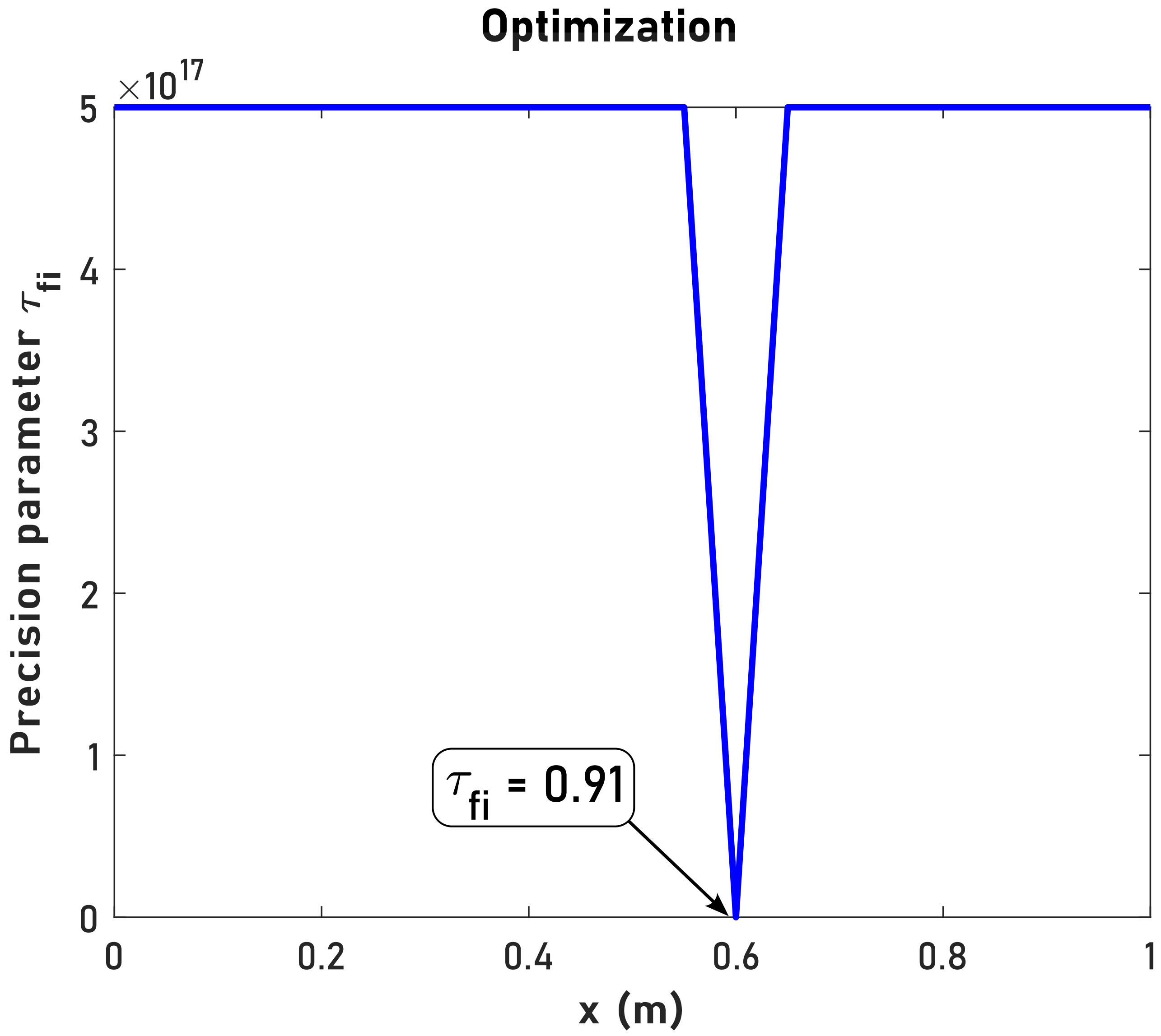


The parameters τ_{fi} and q play a similar role

➥ The larger the value of τ_{fi}, the closer the value of F_i is to 0

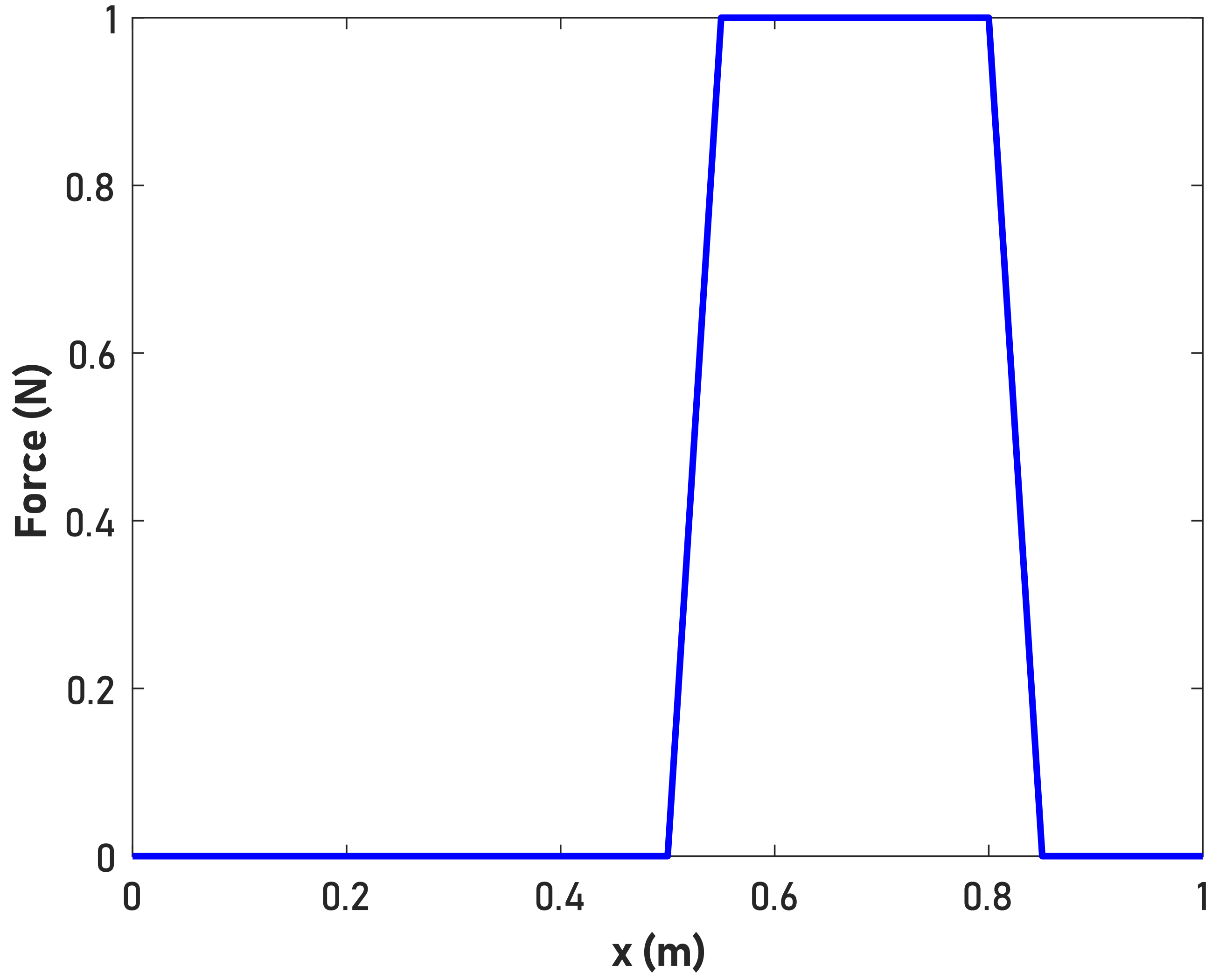
Relevant Vector Regression Why does it work so well ?



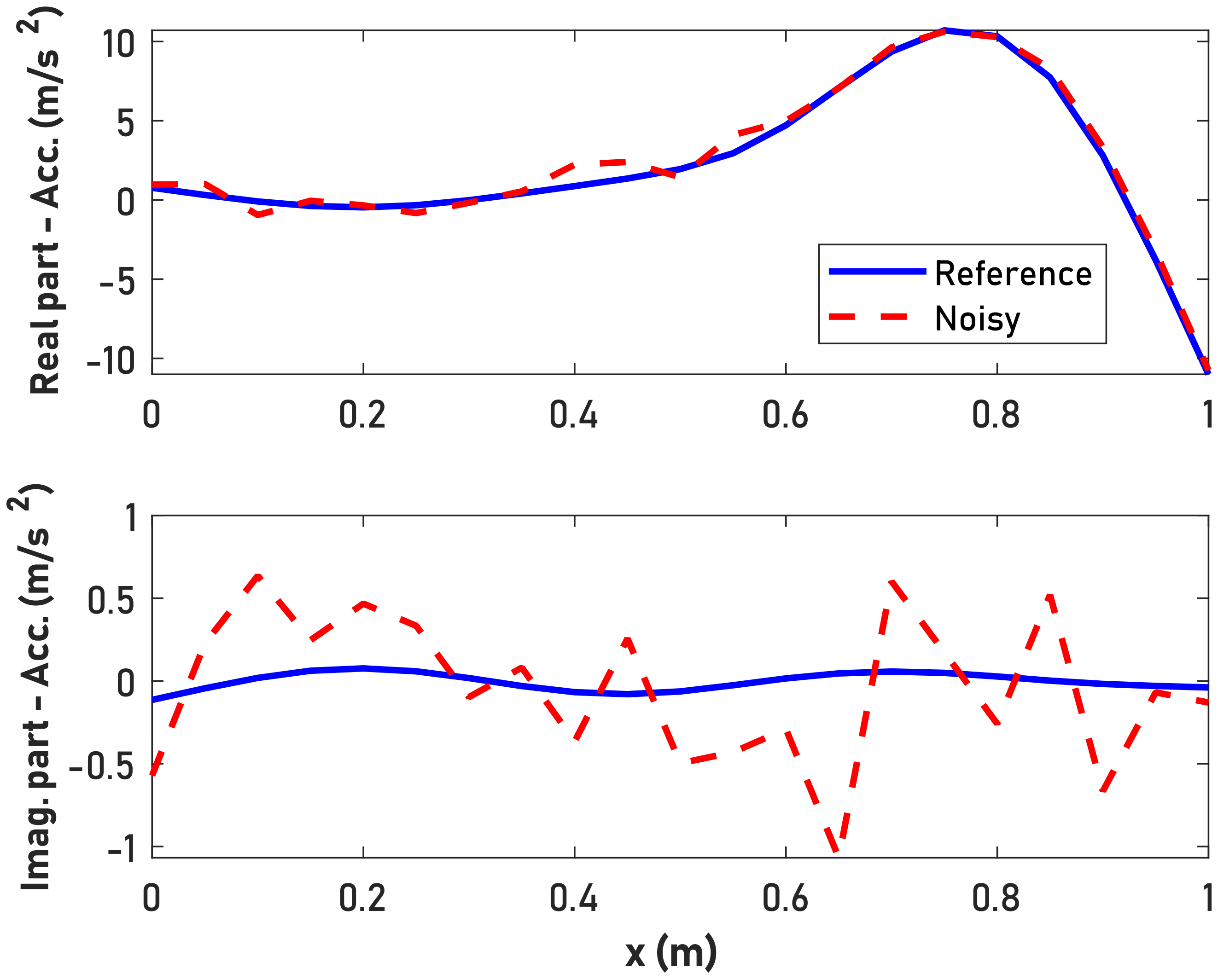


Piecewise constant excitation Objective

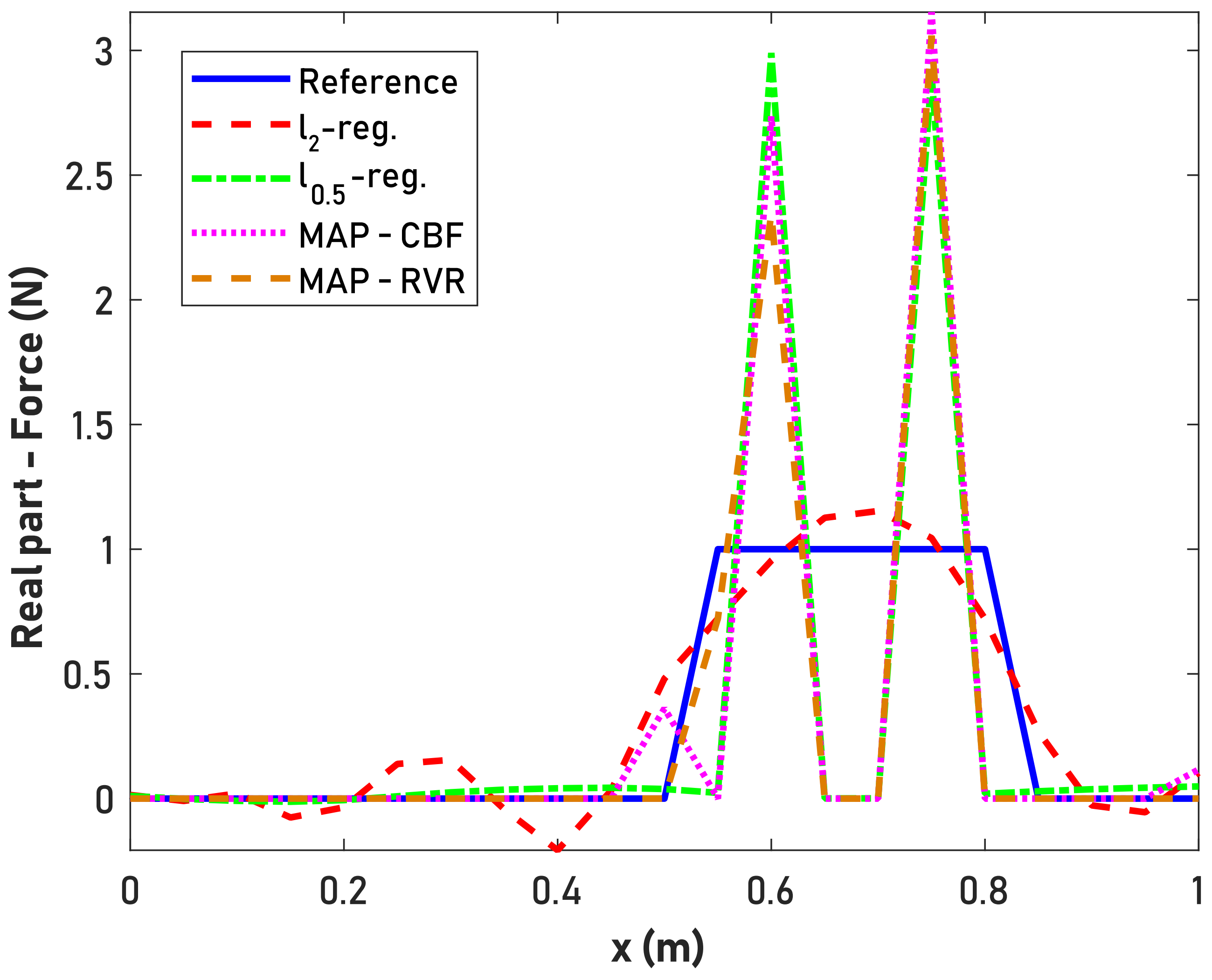
Reconstruct



From



Piecewise constant excitation Naive application

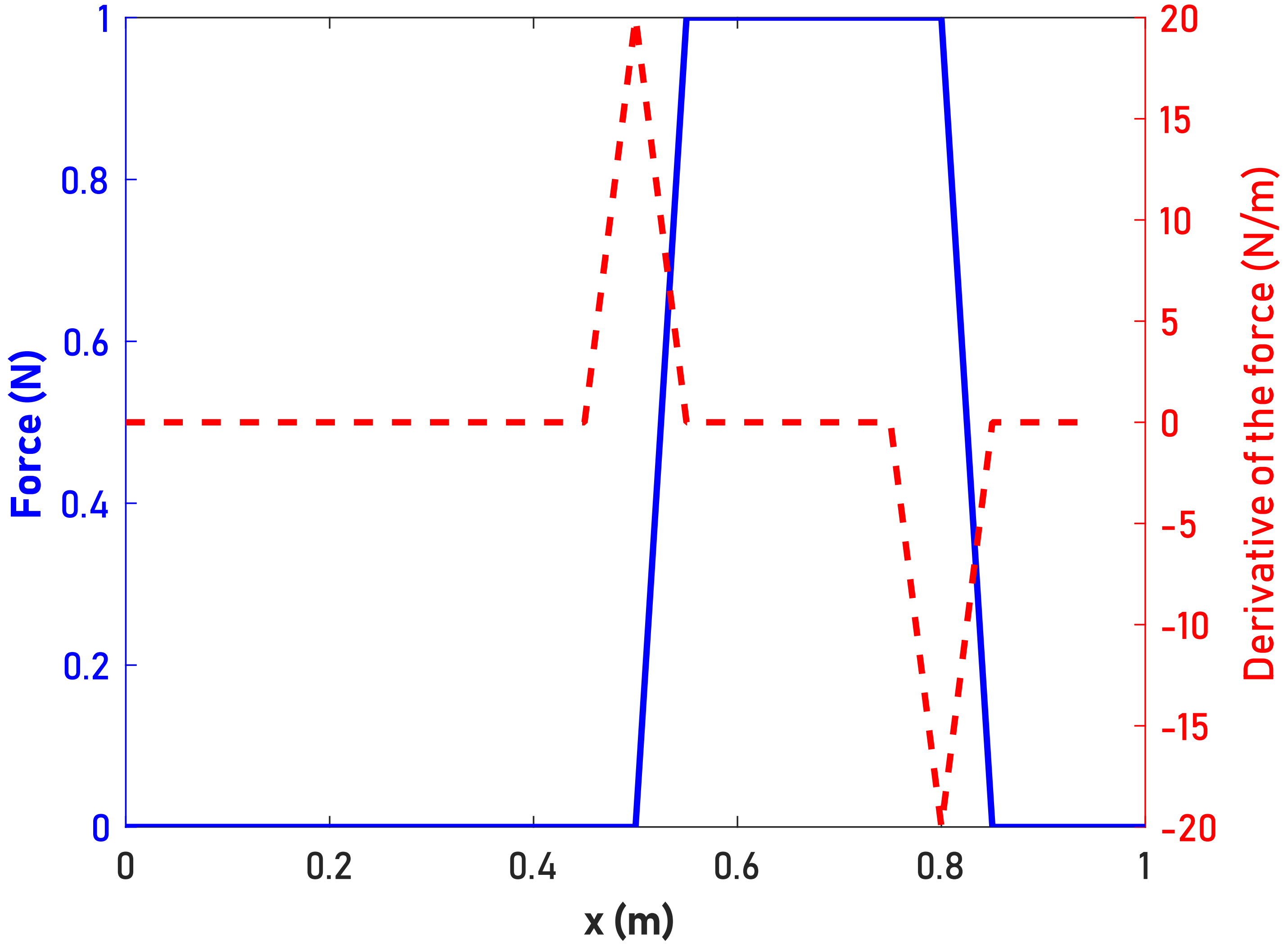


None of the strategies described previously is able to properly reconstruct the excitation field !

What to do ?

Promote piecewise constant solution !

Piecewise constant excitation Intuition



The first derivative of the excitation field is sparse

➡ Promote the sparsity of $\frac{\partial \mathbf{F}(x)}{\partial x}$

Piecewise constant excitation **Implementation**

Using the discretized first-order derivative operator $\mathbf{D}$

$$\mathbf{D} = \frac{1}{\Delta x}\begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}_{(M-1)\times M}$$

One has the following prior probability distributions

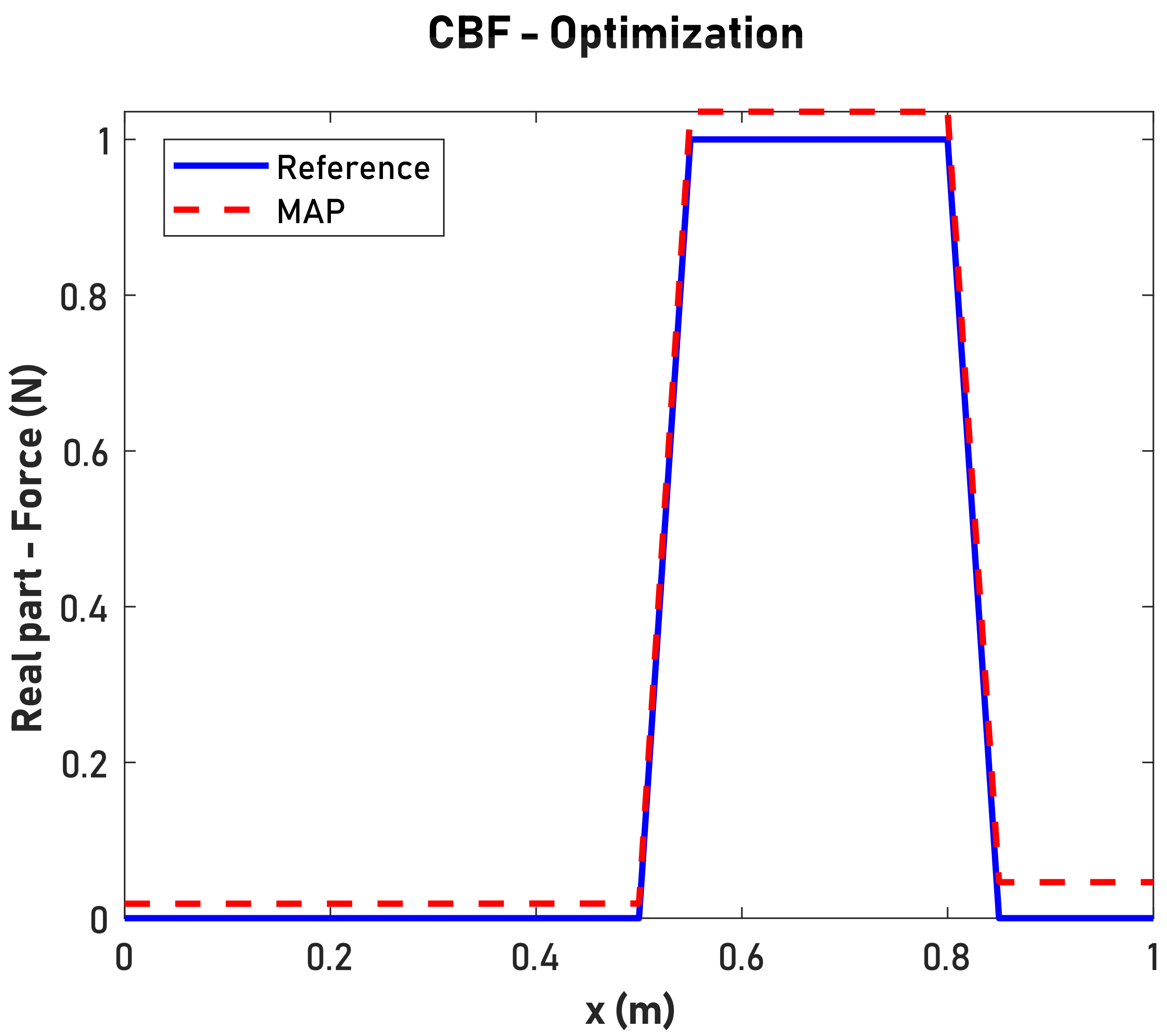
Complete Bayesian formulation

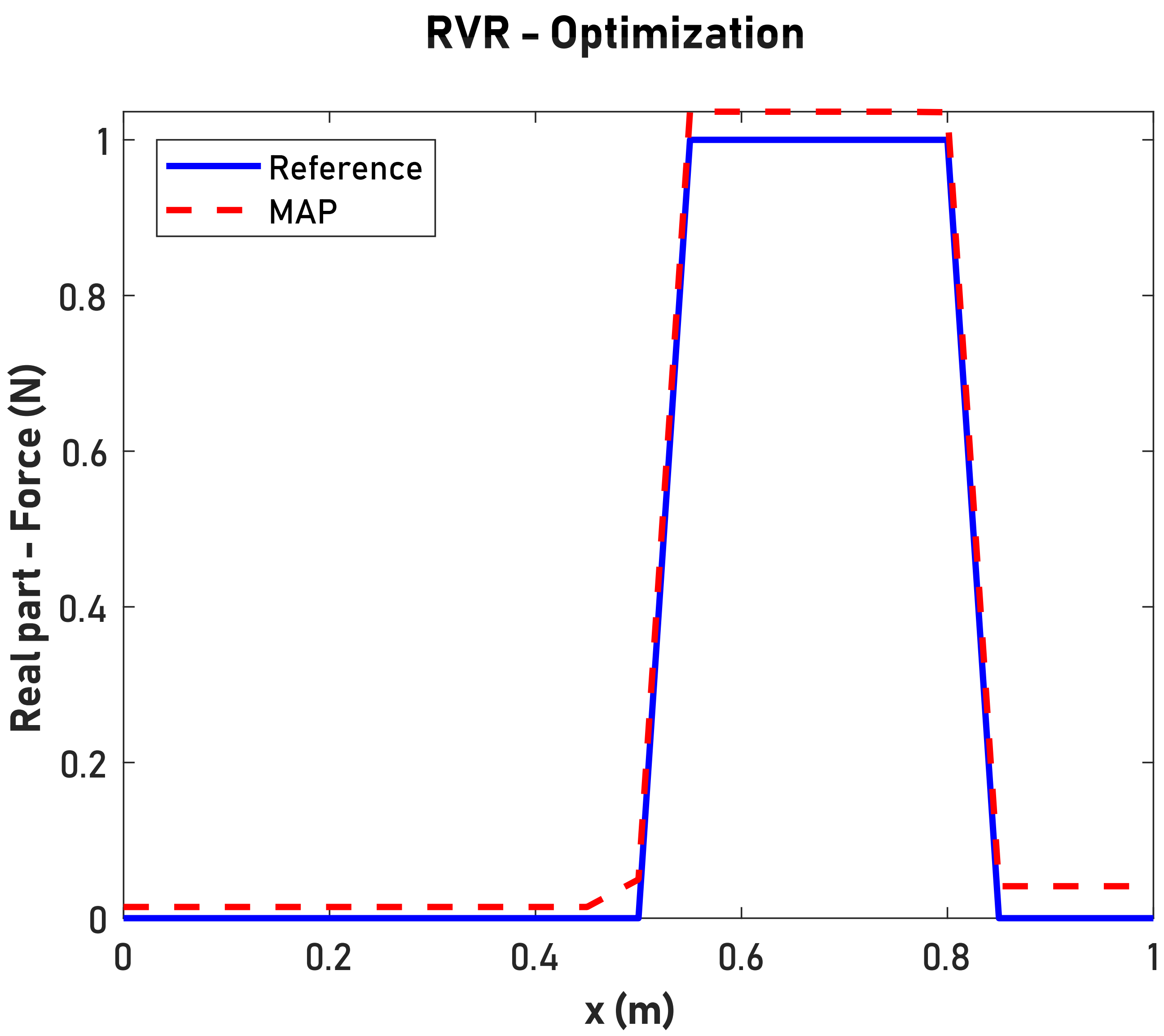
$$p(\mathbf{F}|\tau_f, q) \propto \exp\left(-\tau_f \|\mathbf{D}\mathbf{F}\|_q^q\right)$$

Relevant vector regression

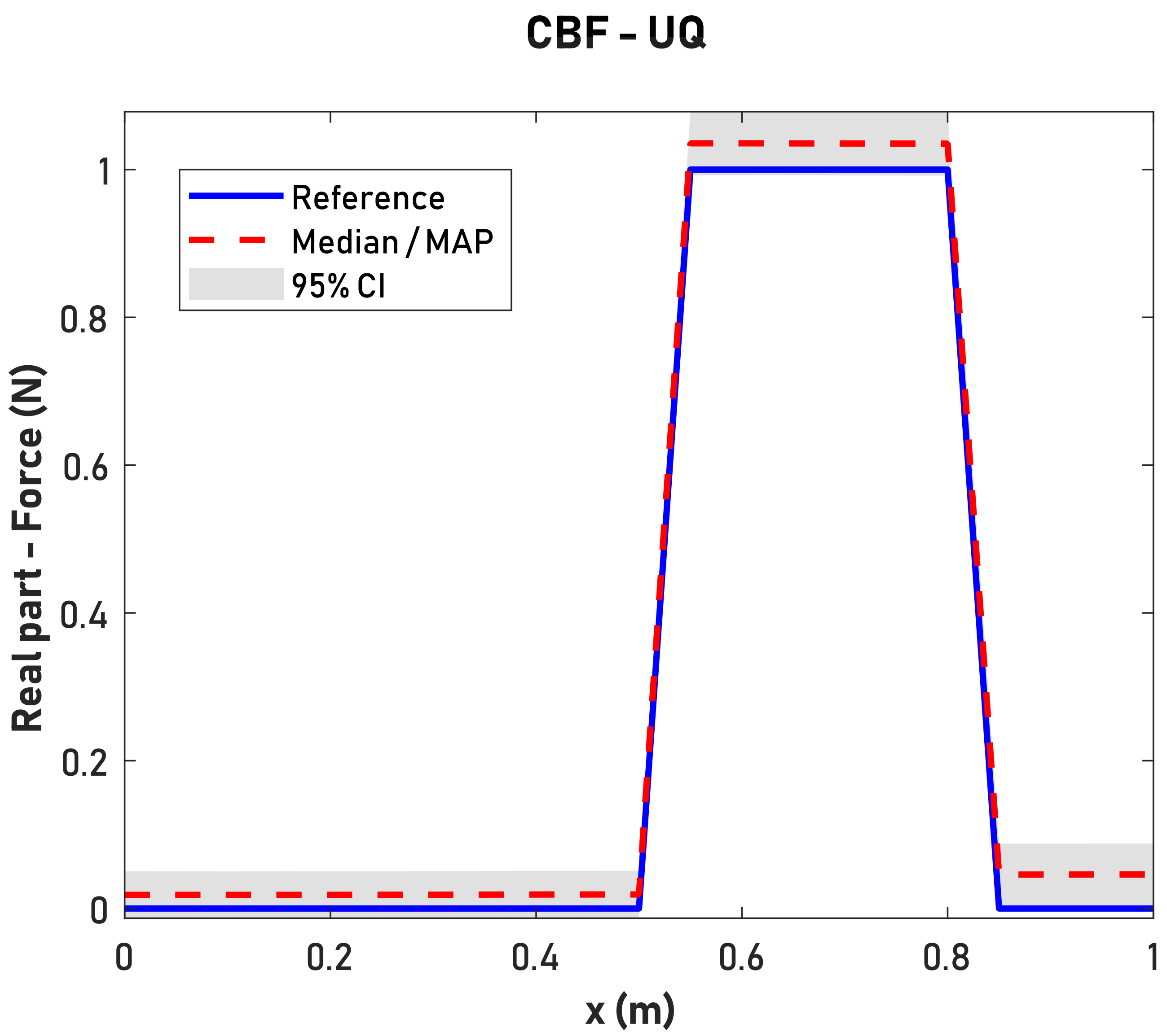
$$p(F_i|\tau_{fj}) \propto \exp\left(-\frac{\tau_{fj}}{2}|D_{ji}F_i|^2\right)$$

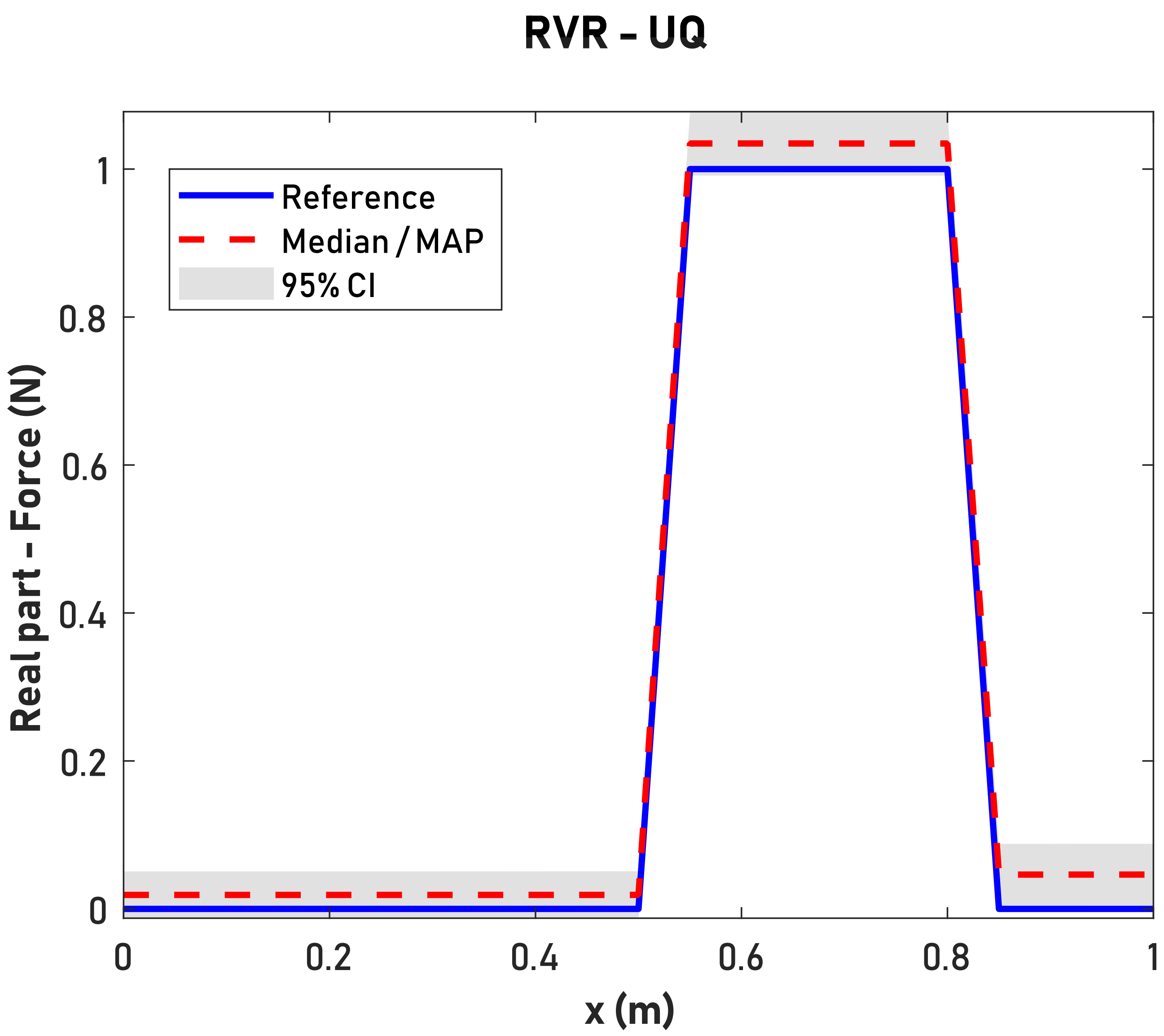
Piecewise constant excitation Application





Piecewise constant excitation Application





Conclusions

- The Bayesian framework provides an efficient and convenient way to combine probabilistic and mechanical data
- It allows exploiting one's prior knowledge of the sources to identify
- It includes an internal mechanism of regularization
- No external procedures are required to infer or optimize all the parameters of the model

Other applications in force reconstruction

- Group regularization – e.g. Identification of external forces and BC on plates
- Mixed-norm regularization – e.g. Identification of space-frequency/time features of excitation sources

Application in other fields

- Image/signal processing (e.g. denoising)
- Acoustics (e.g. fault diagnosis, source reconstruction)
- Material science, Structural mechanics (e.g. parameter estimation, OMA, cracks detection)
- Computer science (e.g. neural networks, bayesian programming)
- Thermal science, Econometrics, Epidemiology, ...

Only the sky is the limit !

Or, maybe, the quantity/quality of available data,
the complexity of the problem,
the computing power/resources, ...

le cnam

Force reconstruction

A Bayesian perspective

https://github.com/maucejo/MOIRA_Workshop_BFR

Well-posed problem in the sense of Hadamard (1902)

- A solution exist
- The solution is unique
- The solution changes continuously with changes in the data

Back to presentation

Well-posed problem in the sense of Hadamard (1902)

✓A solution exist

✓The solution is unique

✗ The solution changes continuously with changes in the data

Back to presentation

Well-posed problem in the sense of Hadamard (1902)

✓A solution exist

✓The solution is unique

✗ The solution changes continuously with changes in the data

➥ The problem considered in this lecture is ill-posed

Back to presentation

ℓ_q-regularization Filter factor analysis at convergence

$$\widehat{\mathbf{F}} = \sum_{i=1}^{21} f_i \frac{\mathbf{v}_i \mathbf{u}_i^H \mathbf{X}}{\sigma_i} \quad \text{with} \quad f_i = \frac{\gamma_i^2}{\gamma_i^2 + \lambda}$$

where γ_i are the singular values of $(\mathbf{H}, \mathbf{L})$ and σ_i are the singular values of $\mathbf{H}$

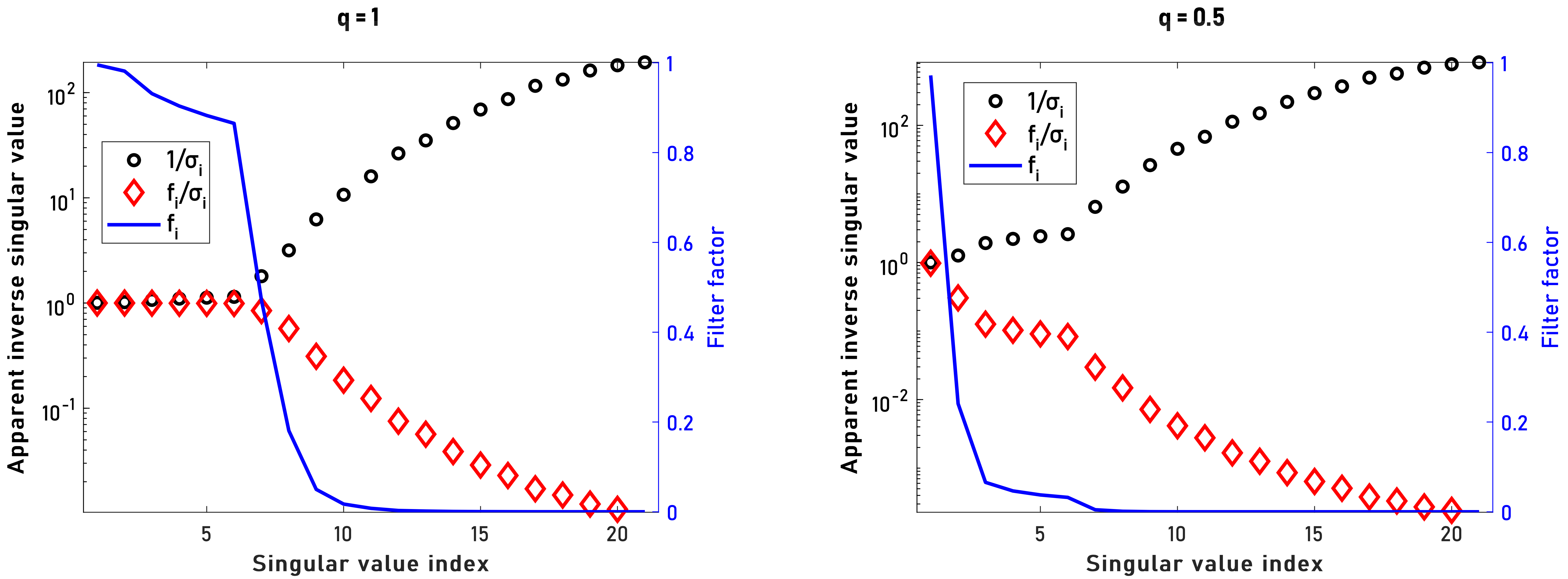
Generalized SVD

$$\mathbf{H} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{Y}^H \quad \text{and} \quad \mathbf{L} = \mathbf{V}\boldsymbol{\Omega}\mathbf{Y}^H$$

Properties of GSVD

$$\boldsymbol{\Sigma}^H\boldsymbol{\Sigma} + \boldsymbol{\Omega}^H\boldsymbol{\Omega} = \mathbf{I} \quad \text{and} \quad \gamma_i = \frac{\sigma_i}{\omega_i}$$

ℓ_q-regularization Filter factor analysis at convergence



Back to presentation

Properties of Gaussian distributions Marginal and Conditional distributions

Let's consider two random vectors, $\mathbf{x}$ and $\mathbf{y}$, such that

$$p(\mathbf{x}) = \mathcal{N}_c(\mathbf{x}|\boldsymbol{\mu}_\mathbf{x}, \boldsymbol{\Sigma}_\mathbf{x}) \text{ and } p(\mathbf{y}|\mathbf{x}) = \mathcal{N}_c(\mathbf{y}|\mathbf{A}\mathbf{x} + \mathbf{b}, \boldsymbol{\Sigma}_\mathbf{y})$$

From these distributions, the marginal and conditional distributions, $p(\mathbf{y})$ and $p(\mathbf{x}|\mathbf{y})$ are given by

$$p(\mathbf{y}) = \mathcal{N}_c\left(\mathbf{y}|\mathbf{A}\boldsymbol{\mu}_\mathbf{x} + \mathbf{b}, \mathbf{A}\boldsymbol{\Sigma}_\mathbf{x}\mathbf{A}^H + \boldsymbol{\Sigma}_\mathbf{y}\right)$$
$$p(\mathbf{x}|\mathbf{y}) = \mathcal{N}_c\left(\mathbf{x}|\boldsymbol{\Sigma}\{\mathbf{A}^H\boldsymbol{\Sigma}_\mathbf{y}^{-1}(\mathbf{y} - \mathbf{b}) + \boldsymbol{\Sigma}_\mathbf{x}^{-1}\boldsymbol{\mu}_\mathbf{x}\}, \boldsymbol{\Sigma}\right)$$

with $\boldsymbol{\Sigma} = \left(\mathbf{A}^H\boldsymbol{\Sigma}_\mathbf{y}^{-1}\mathbf{A} + \boldsymbol{\Sigma}_\mathbf{x}^{-1}\right)^{-1}$

Drawing samples from multivariate Gaussian distribution

Let's consider a random Gaussian vector $\mathbf{x}$ such that

$$p(\mathbf{x}) = \mathcal{N}_c(\mathbf{x}|\boldsymbol{\mu}_{\mathbf{x}}, \boldsymbol{\Sigma}_{\mathbf{x}})$$

By assuming that $\boldsymbol{\Sigma}_{\mathbf{x}} = \mathbf{S}\mathbf{S}^H$, one has

$$\begin{aligned} \exp\left[-(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{x}})^H \boldsymbol{\Sigma}_{\mathbf{x}}^{-1}(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{x}})\right] &= \exp\left[-\{\mathbf{S}^{-1}(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{x}})\}^H\{\mathbf{S}^{-1}(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{x}})\}\right] \\ &= \exp\left[-\mathbf{z}^H\mathbf{z}\right] \end{aligned}$$

where $\mathbf{z} = \mathbf{S}^{-1}(\mathbf{x}-\boldsymbol{\mu}_{\mathbf{x}})$ and $\mathbf{z} \sim \mathcal{N}_c(\mathbf{z}|\mathbf{0}, \mathbf{I})$

Consequently, to draw samples from a multivariate Gaussian distribution with mean $\boldsymbol{\mu}_{\mathbf{x}}$ and covariance matrix $\boldsymbol{\Sigma}_{\mathbf{x}}$, it is enough to compute

$$\mathbf{x}^{(k)} = \boldsymbol{\mu}_{\mathbf{x}} + \mathbf{S}\,\mathbf{z}^{(k)} \quad \text{with} \quad \mathbf{S}\mathbf{S}^H = \boldsymbol{\Sigma}_{\mathbf{x}} \quad \text{and} \quad \mathbf{z}^{(k)} \sim \mathcal{N}_c(\mathbf{z}^{(k)}|\mathbf{0}, \mathbf{I})$$

Back to presentation

Calculation of τ_n and τ_f

By using the Bayes' rule, the conditional distribution $p(\tau_n, \tau_f|\mathbf{X})$ is expressed as

$$p(\tau_n, \tau_f|\mathbf{X}) \propto p(\mathbf{X}|\tau_n, \tau_f)\, p(\tau_n)\, p(\tau_f)$$

Assuming that $p(\tau_n) = p(\tau_f) \propto 1$, one has

$$p(\tau_n, \tau_f|\mathbf{X}) \propto p(\mathbf{X}|\tau_n, \tau_f) = \int_{\mathbf{F}} p(\mathbf{X}|\mathbf{F}, \tau_n)\, p(\mathbf{F}|\mathbf{W}, \tau_f) d\mathbf{F}$$

Using the fact that all the conditional distributions are Gaussian, one establishes that

$$p(\tau_n, \tau_f|\mathbf{X}) \propto \mathcal{N}_c(\mathbf{X}|\mathbf{0}, \mathbf{H}\mathbf{W}^{-1}\mathbf{H}^H/\tau_f + \mathbf{I}/\tau_n)$$

The MAP estimate is found by solving

$$(\widehat{\tau}_n, \widehat{\tau}_f) = \underset{(\tau_n, \tau_f)}{\operatorname{argmin}} \; -\log[p(\tau_n, \tau_f|\mathbf{X})]$$

By noting $\lambda = \tau_n / \tau_f$, it comes

$$(\widehat{\tau}_n, \widehat{\tau}_f) = \underset{(\tau_n, \tau_f)}{\operatorname{argmin}} \, \tau_f \, \mathbf{X}^H \left(\mathbf{H}\mathbf{W}^{-1}\mathbf{H}^H + \lambda \mathbf{I}\right)^{-1} \mathbf{X} - N \log \tau_f + \log \left|\mathbf{H}\mathbf{W}^{-1}\mathbf{H}^H + \lambda \mathbf{I}\right|$$

By applying the first-order optimality condition, one finds

$$\widehat{\tau}_f = \frac{N}{\mathbf{X}^H \left(\mathbf{H}\mathbf{W}^{-1}\mathbf{H}^H + \lambda \mathbf{I}\right)^{-1} \mathbf{X}}$$

Back to presentation